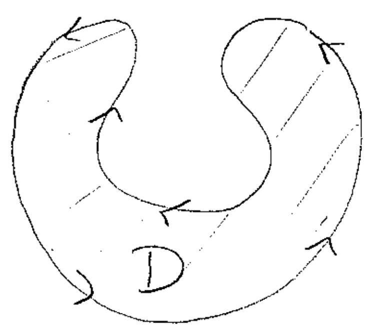
Lecture 40 Green's Theorem (§8.1)

HW: On web.

Next time: More on Green's Theorem.

• C a closed curve bounding a region D in $\mathbb{R}^2$, oriented so D is to the left as you go around.



• $\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2$ a vector field given by $\vec{F}(x,y) = (F_1(x,y), F_2(x,y))$

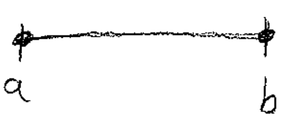
Green's Thm:

$$\int_C \vec{F} \cdot ds = \iint_D \underbrace{\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)}_{\text{[scalar curl.]}} dx\,dy$$

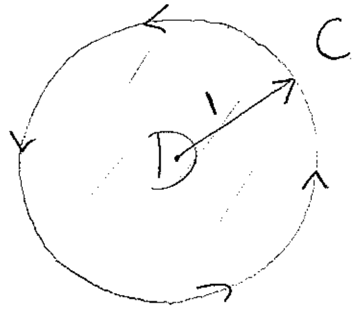
At first glance, this seems rather mysterious: how can an integral over the whole region depend only on $\vec{F}$ at the boundary.

Compare:

$$f(b) - f(a) = \int_a^b f'(x)\,dx.$$

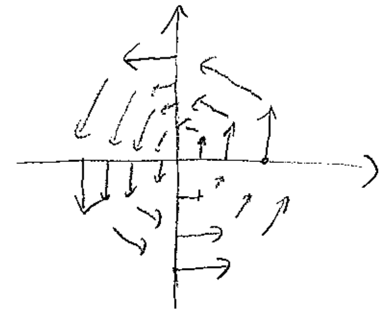


Ex:



$$\vec{F} = \tfrac{1}{2}(-y, x)$$

$$C(t) = (\cos t, \sin t)$$
$$0 \le t \le 2\pi$$



$$\int_C F \cdot ds = \int_0^{2\pi} F(c(t)) \cdot c'(t)\, dt =$$

$$\int_0^{2\pi} \tfrac{1}{2}(-\sin t, \cos t) \cdot (-\sin, \cos t)\, dt = \int_0^{2\pi} \tfrac{1}{2}\, dt = \boxed{\pi}$$

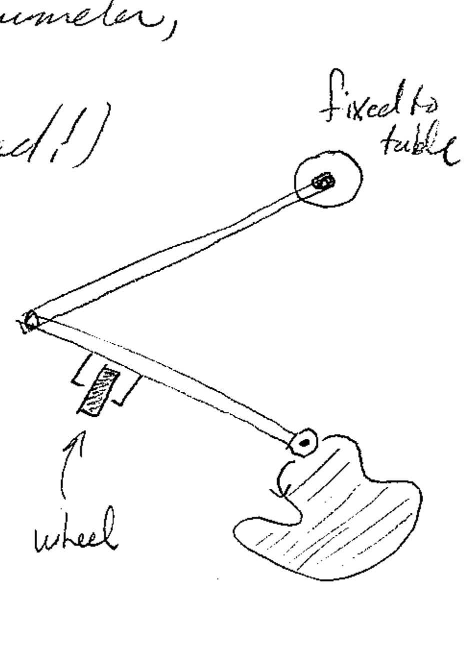
The Same!

$$\iint_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dA = \iint_D \tfrac{1}{2}(1+1)\, dA = \iint_D 1\, dA = \boxed{\pi}$$

Special Case: $\vec{F} = \tfrac{1}{2}(-y, x)$ for any region

Then $\quad \text{Area}(D) = \int_C \vec{F} \cdot ds$

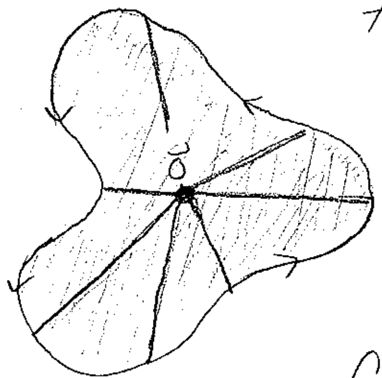
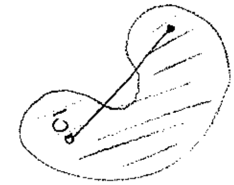
This fact is actually used in a planimeter, a mechanical device (still manufactured!) used to measure areas.



This was perfected by a Swiss mathematician in 1854...

Here's why Green's Theorem works in this special case.

Suppose $\vec{0}$ is in D and D is "star shaped,"

that is, like  and not 

C

The integral

$$\int_C \vec{F} \cdot ds = \int_a^b \vec{F}(c(t)) \cdot c'(t)\, dt$$

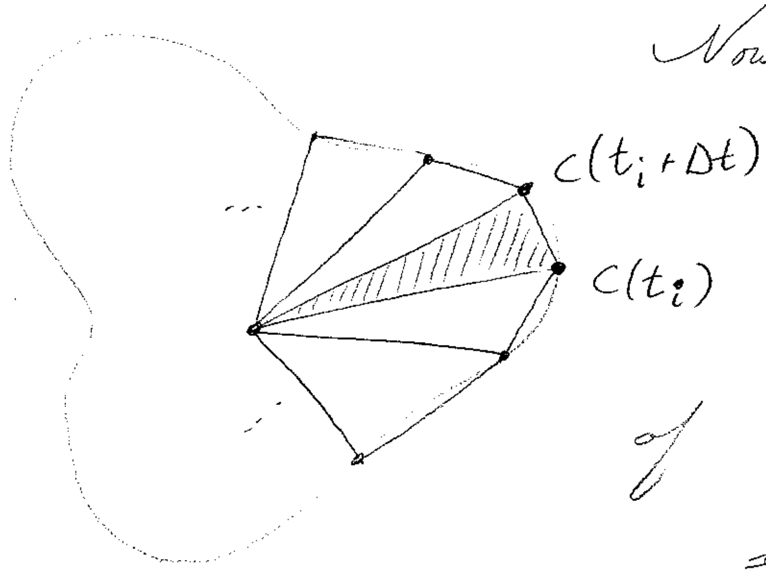
$$= \int_a^b \frac{1}{2}(-c_2(t), c_1(t)) \cdot (c_1'(t), c_2'(t))\, dt$$

$$= \int_a^b \frac{1}{2}\left(c_1(t)c_2'(t) - c_2(t)c_1'(t)\right) dt$$

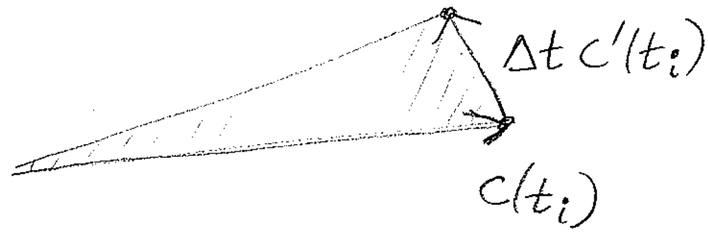
So a typical term in a Riemann sum approximating this integral is

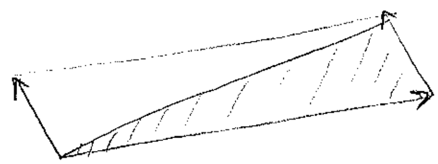
$$\circledast \quad \frac{1}{2}\left(c_1(t_i)c_2'(t_i) - c_2(t_i)c_1'(t_i)\right)\Delta t$$

Well, now let's put this aside and think about finding the area



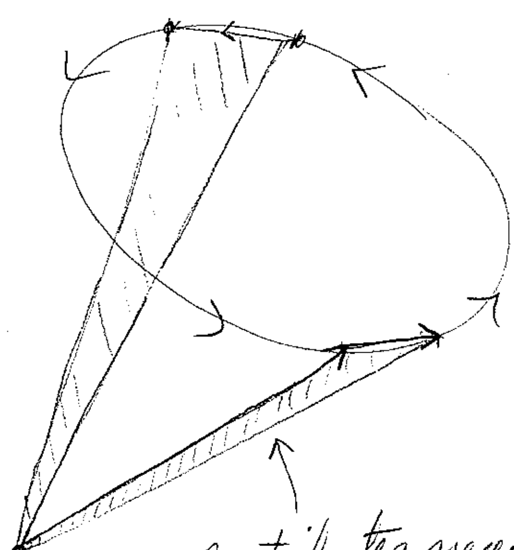
Now the area of the wedge shown is $\approx$ the area of



$= \frac{1}{2}$ area of  $= \frac{1}{2} \det \begin{pmatrix} c_1(t_i) & \Delta t\, c_1'(t_i) \\ c_2(t_i) & \Delta t\, c_2'(t_i) \end{pmatrix}$

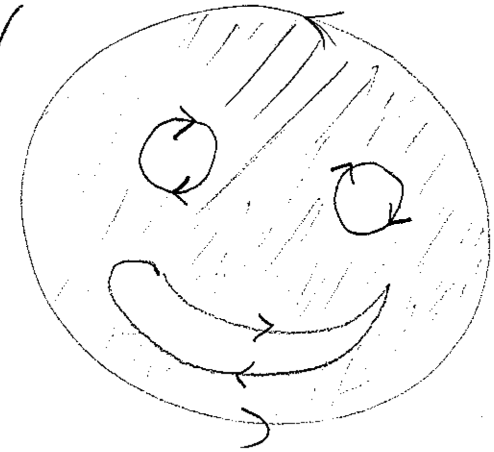
$= \otimes$. Thus, the area of D can be approximated by a Riemann sum for $\int_C \vec{F} \cdot ds$ and in the limit $\int_C \vec{F} \cdot ds = \text{Area}(D)$!

What if $\vec{0}$ is not in D? Then we get both positive and negative contributions, and the too large positive contributions are cancelled out by the negative bits.



contributes negative area.

Green's Theorem applies to regions with multiple boundries, you just have to orient them in the correct way — the region should always be to your left



$$\int_{\partial D} F\cdot ds = \sum_{\substack{\text{boundary}\\ \text{curves } C_i}} \int_{C_i} \vec{F}\cdot ds = \iint_D \left(\frac{\partial F_2}{\partial y} - \frac{\partial F_1}{\partial x}\right) dA$$

Alternate Notation:

$C:[a,b] \longrightarrow \mathbb{R}^2$ a curve, $\vec{F}=(F_1,F_2)$ a vector field

$$\int_C F\cdot ds = \int_C F_1\,dx + F_2\,dy$$

This is because if we write $c(t)=(x(t),y(t))$

Then

$$F\cdot ds = F(x(t),y(t))\cdot \overset{(x'(t),\,y'(t))}{\overset{\shortparallel}{C'(t)}} =$$

$$F_1(x(t),y(t))\,\underbrace{x'(t)\,dt}_{dx} + F_2(x(t),y(t))\,\underbrace{y'(t)\,dt}_{dy}$$

Book writes $\vec{F} = (P, Q)$ and so Green's Thm becomes

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy.$$

Historical notes: Vector Calculus was discovered in the 19th century, and was perfected by Gibbs.