Lecture 48: Stokes Theorem and Conservative Vector Fields
~ and ~
Maxwell's Equations

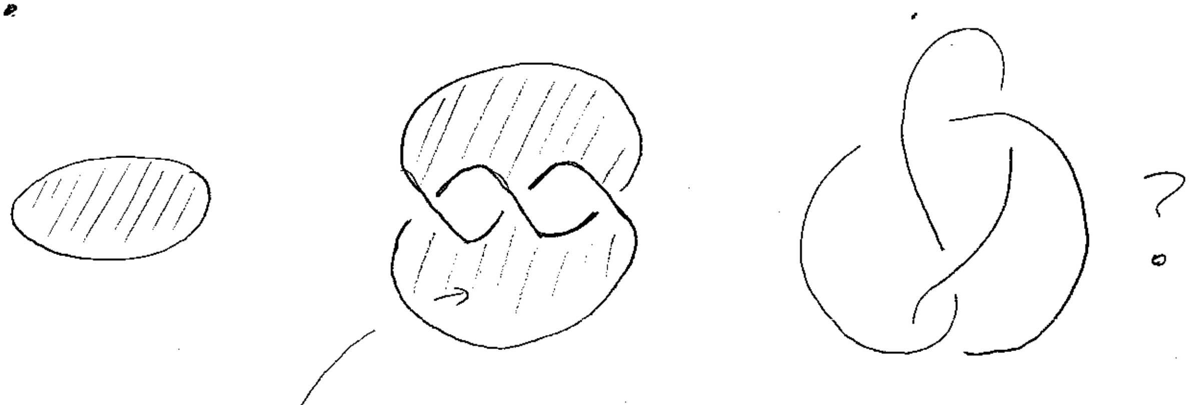
$\vec{F}$ a vector field on U in $\mathbb{R}^3$. If U is simply connected, then $\vec{F}$ is conservative if and only if curl $\vec{F} = \vec{0}$ everywhere.

__Idea__: Suppose curl $\vec{F} = \vec{0}$ everywhere. Let C be a closed path in U. If C is the boundary of an orientable surface S contained in U,

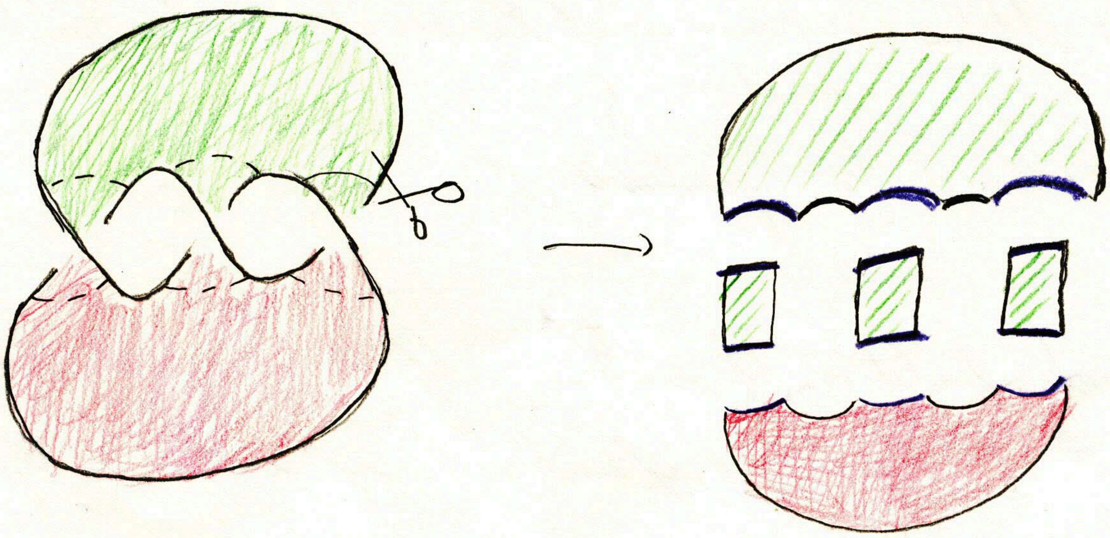
then $$\int_C \vec{F} \cdot ds = \iint_S (\text{curl } \vec{F}) \cdot \vec{n}\, dS = \iint_S 0\, dS = 0$$

as is required for a conservative vector field.

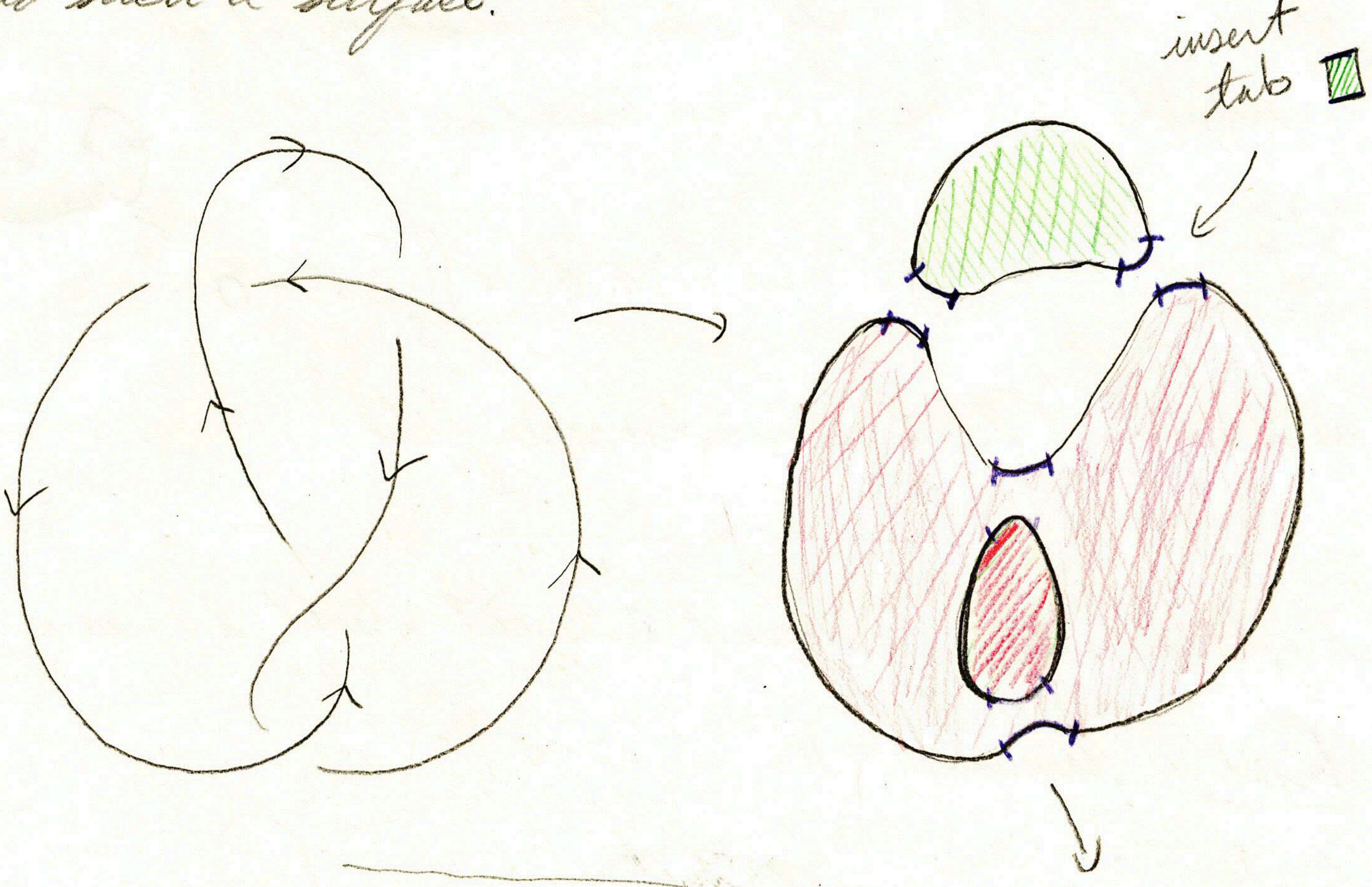
Might as assume C doesn't intersect itself, i.e. is a __knot__. Assume $U = \mathbb{R}^3$.



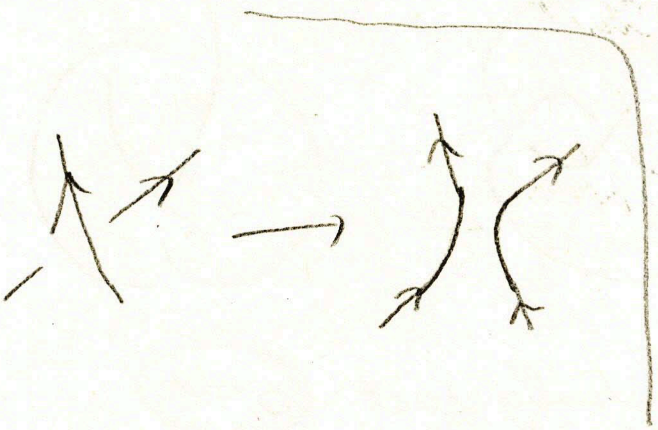
This surface is actually just 

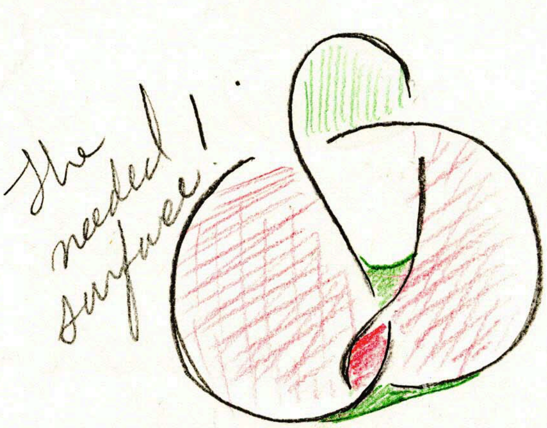


Here's an example of using Seifert's Algorith to find such a surface.



Where





For more, see Colin Adams, <u>Why Knot?</u> or <u>The Knot Book</u>.

<u>Maxwell's Equations</u>:

$\vec{E}(x,y,z,t)$ — Electric Field

$\vec{B}(x,y,z,t)$ — Magnetic Field

$\rho(x,y,z,t)$ — charge density

<u>Gauss's Law</u>:

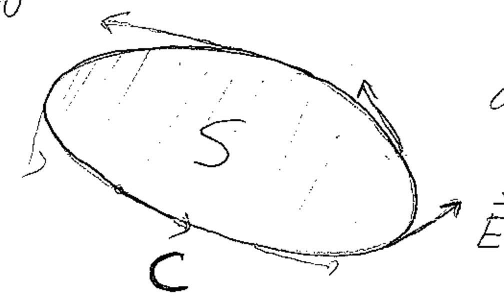
$$\operatorname{div} \vec{E} = \frac{\rho}{\varepsilon_0} \qquad \iint_{\partial R} \vec{E}\cdot\vec{n} = \iiint_R \rho \, dV$$

<u>Gauss's Law for magnetic fields</u>:

$$\operatorname{div} \vec{B} = 0 \qquad \iint_S (\vec{B}\cdot\vec{n})\, dA = 0$$

[No magnetic monopoles.]

<u>Faraday's Law of induction</u>: A changing magnetic field induces a current in a loop of wire.

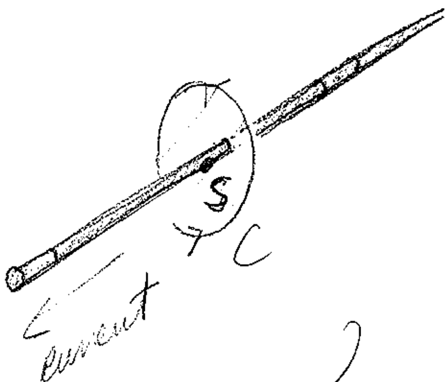


$$\int_C \vec{E}\cdot ds = -\frac{\partial}{\partial t}\iint_S (\vec{B}\cdot\vec{n})\, dS$$

Now $\int_C \vec{E}\cdot ds = \iint_S (\text{curl}\,\vec{E})\cdot\vec{n}\,dA$, and so

$$\boxed{\text{curl}\,\vec{E} = -\frac{\partial\vec{B}}{\partial t}}$$

Ampére's Circuital Law:



$$\int_C \vec{B}\cdot ds = \mu_0 \iint_S (\vec{J}\cdot\vec{n})\,dA + \varepsilon_0\mu_0 \frac{\partial}{\partial t}\iint_S (\vec{E}\cdot\vec{n})\,dA$$

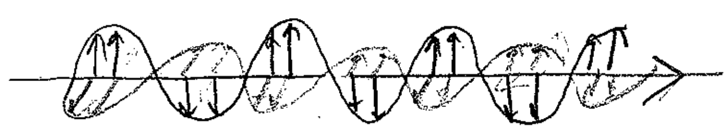
Now again $\int_C \vec{B}\cdot ds = \iint_S (\text{curl}\,\vec{B})\cdot\vec{n}\,dA$

and so

$$\boxed{\text{curl}\,\vec{B} = \mu_0\vec{J} + \varepsilon_0\mu_0 \frac{\partial\vec{E}}{\partial t}}$$

speed of light
$c = \frac{1}{\sqrt{\varepsilon_0\mu_0}}$

And There Was Light:



$$\vec{E} = (0,\ c\,B_{max}\cos 2\pi(\omega t - x/\lambda),\ 0)$$

$$\vec{B} = (0,\ 0,\ B_{max}\cos 2\pi(\omega t - x/\lambda))$$

$$c = \omega\lambda.$$