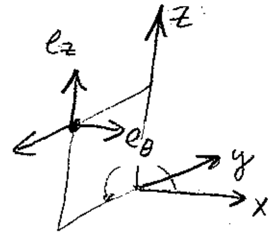
Lecture 47: Stokes Theorem and Conservative Vector Fields (112)
(§8.3 and 5.4)

HW: Handout.

Next time: More on this + §8.5 on Maxwell's Equations

Last time:



$$F = F_r e_r + F_\theta e_\theta + F_z e_z$$

$$\text{curl } \vec{F} = \frac{1}{r} \det \begin{pmatrix} e_r & r e_\theta & e_z \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial z \\ F_r & r e_\theta & F_z \end{pmatrix}$$

See "div, grad curl and all that" for details;
see pages 298-301 of our text for such formulas.

Recall: A vector field $\vec{F}: \mathbb{R}^n \to \mathbb{R}^n$ is conservative if

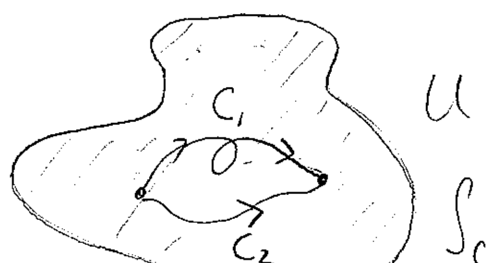
$$\vec{F} = \text{grad } f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots\right) \quad \text{for some } f: \mathbb{R}^n \to \mathbb{R}$$

[Query: What do we about line integrals of conservative vector fields?]

Theorem: $\vec{F}$ a vector field on (open) region U in $\mathbb{R}^n$.
The following are equivalent

a) $\vec{F}$ is conservative.

b) $\vec{F}$ is path independent



$$\int_{C_1} \vec{F} \cdot ds = \int_{C_2} \vec{F} \cdot ds$$

c) For every closed path C, $\int_C F \cdot ds = 0$.

In $\mathbb{R}^2$, any of (a–c) imply that s. curl $\vec{F} = 0$ everywhere.
In $\mathbb{R}^3$, any of (a–c) imply that curl $F = \vec{0}$ everywhere.

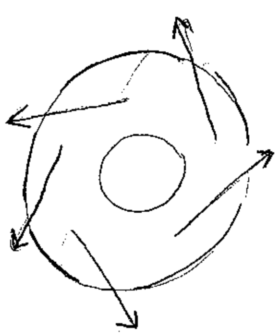
<u>Check:</u> $\vec{F} = \operatorname{grad} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

$$\operatorname{curl} \vec{F} = \det \begin{pmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial f}{\partial x} & \partial f/\partial y & \partial f/\partial z \end{pmatrix}$$

$$= (0, 0, 0)$$

[<u>Query</u>: Why didn't I throw this curl condition into the theorem?]

<u>Ex</u>: $U = \{1 < \|\vec{x}\| < 2\}$ in $\mathbb{R}^2$



$\vec{F} = \frac{1}{x^2+y^2}(-y, x)$ has s. curl $= 0$

but is not conservative.

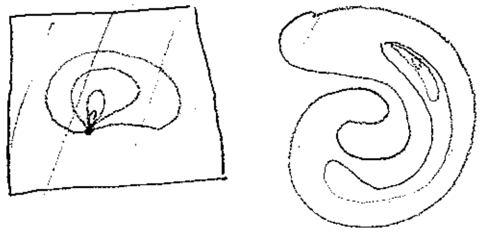
<u>Def</u>: A region U is <u>connected</u> if every two points can be joined by a path

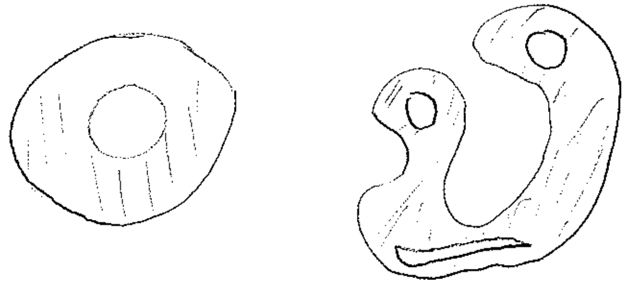
<u>Def</u>: A connected region is <u>simply connected</u> if every closed curve in U can be shrunk to a point (without breaking it) without leaving U.

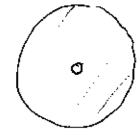
[<u>Think</u>: simply connected = "no holes" for $\mathbb{R}^2$]

Simply connected | Not simply connected

$\mathbb{R}^2$:

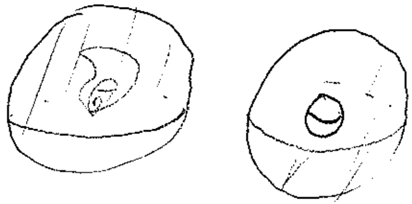




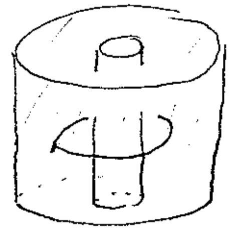


center removed.

$\mathbb{R}^3$



$1 < \|x\| < 2$

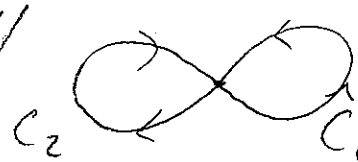


Thm: U region in $\mathbb{R}^2$ or $\mathbb{R}^3$ which is simply connected.
Then $\vec{F}: U \to \mathbb{R}^3$ is conservative if and only if $\begin{cases} \text{scurl } \vec{F} = 0 & (\mathbb{R}^2) \\ \text{curl } \vec{F} = \vec{0} & (\mathbb{R}^3) \end{cases}$
everywhere.

Why this works in $\mathbb{R}^2$: Suppose scurl $\vec{F} = 0$

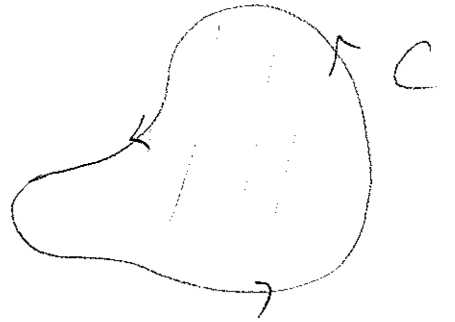
Let C be a closed curve in U; need to check $\int_C \vec{F} \cdot ds = 0$

If C intersects itself



then we can break it into pieces and $\int_C \vec{F} \cdot ds = \int_{C_1} \vec{F} \cdot ds + \int_{C_2} \vec{F} \cdot ds$

So we can just consider "simple" curves which don't intersect themselves.



Now C bounds a region D (a disc by the Jordan Curve Theorem)

D must be contained inside U or U has a hole. Thus

$$\int_C \vec{F}\cdot ds = \iint_D \text{scurl}\, \vec{F}\, dA = 0$$

So $\vec{F}$ is conservative.

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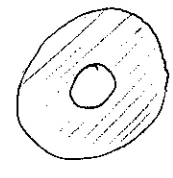
Discuss what goes wrong for the initial region 

<u>Idea for $\mathbb{R}^3$:</u> Suppose curl $\vec{F} = \vec{0}$ everywhere, C a closed path. Want $\int_C \vec{F}\cdot ds = 0$. If C is the boundary of some surface S in U, then

$$\int_C \vec{F}\cdot ds \underset{\text{Stokes Thm}}{=} \iint_S (\text{curl}\,\vec{F})\cdot\vec{n}\, dA = \iint_S 0\, dA = 0.$$

as desired. <u>Next time</u>: Why are there always such surfaces?