Lecture 46: Application of Stokes Theorem

HW: 8.3 #13. Verify Stokes theorem for 5 + 21.   §4.8 #17

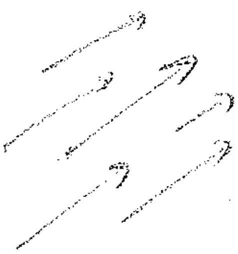
Next time:

Stokes Theorem: S a surface in $\mathbb{R}^3$ with normal vector field $\vec{n}$. $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$ a vector field. Then

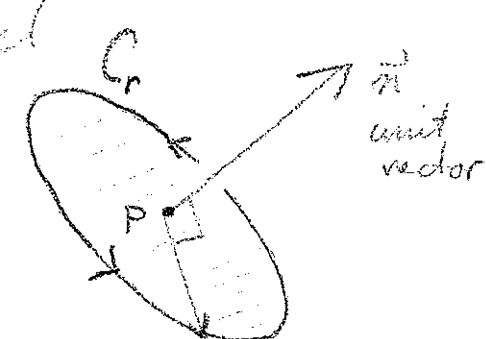
$$\int_{\partial S} \vec{F} \cdot ds = \iint_S (\text{curl}\,\vec{F}) \cdot \vec{n}\, dA$$

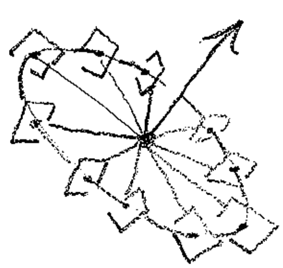
Here ∂S can have multiple boundary curves as long as they are oriented correctly.

Understanding Curl: $\vec{F}$ = velocity of flowing fluid



Consider a paddle wheel with small paddles out on the boundary of D_r



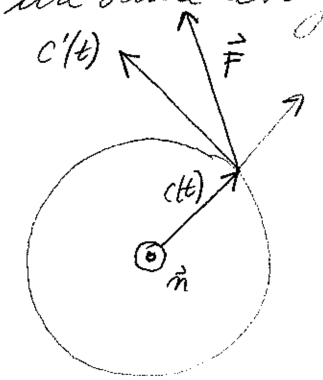


D_r = disc of radius r, $\perp$ to $\vec{n}$.

<u>Key</u>: The rate at which this wheel rotates is

$$\omega = \frac{1}{2\pi r^2} \int_{C_r} \vec{F} \cdot ds$$

To see why, suppose the wheel rotates with angular speed ω, e.g. $c(t) = (r\cos\omega t, r\sin\omega t, 0)$ if $\vec{n} = (0,0,1)$. If the tangential component of $\vec{F}$ is the same everywhere, we would have



$$c'(t) = \mathrm{Proj}_{c'(t)} \vec{F} = \vec{F} \cdot \vec{u}$$

where $\vec{u} = \dfrac{c'(t)}{\|c'(t)\|}$ is the unit tangent vector.

In general, the average rotational components should be equal

$$\int_{C_r} \vec{F} \cdot \vec{u} \, ds = \int_{C_r} \vec{F} \cdot ds$$

$$\begin{gathered} \shortparallel \\ \int_{C_r} \|c'(t)\| \, ds = \int_{C_r} \omega r \, ds = 2\pi r^2 \omega \implies \end{gathered}$$

$$c'(t) = (-\omega r \sin\omega t, \omega r\cos\omega t, 0) \qquad \omega = \frac{1}{2\pi r^2} \int_{C_r} \vec{F} \cdot ds$$

Now Stokes Theorem we get

$$\omega = \frac{1}{2\pi r^2}\int_{C_r} F\cdot ds = \frac{1}{2}\overbrace{\frac{1}{\text{Area}(D_r)}\iint_{D_r}(\text{curl}\,F)\cdot\vec{n}\,dA}^{\text{Average of }(\text{curl}\,\vec{F})\cdot\vec{n}}$$

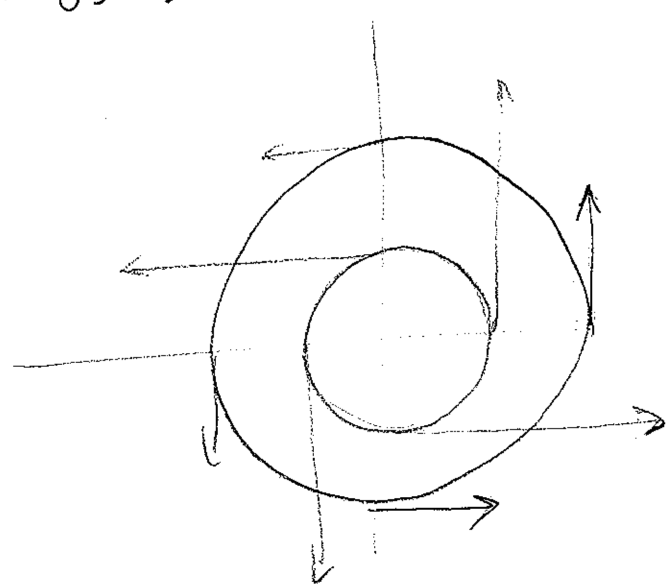
Taking $r \to 0$, get $\omega = \frac{1}{2}(\text{curl}\,F(\vec{p}))\cdot\vec{n}$.

Thus the rate of rotation is largest in the direction of curl $\vec{F}$ and then $\omega = \frac{1}{2}\|\text{curl}\,F(\vec{p})\|$.

Note: Vector fields with curl $\vec{F} = \vec{0}$ are called irrotational.

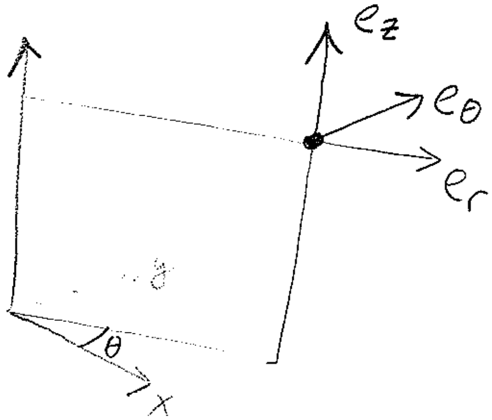
However they can still rotate; experimentally, a typical bathtub draining is "irrotational."

Ex: $F(x,y) = \frac{1}{x^2+y^2}(-y, x)$



Curl and friends in other coordinates:

Cylindrical coordinates



Alternate orthonormal basis at (r, θ, z):

$$e_r = (\cos\theta, \sin\theta, 0)$$
$$e_\theta = (-\sin\theta, \cos\theta, 0)$$
$$e_z = (0, 0, 1)$$

Can express vector fields using these.

E.g. the example above is $\vec{F} = \frac{1}{r} e_\theta$

Suppose $\vec{F} = F_r e_r + F_\theta e_\theta + F_z e_z$ — functions $\mathbb{R}^3 \to \mathbb{R}$

then

$$\operatorname{curl} \vec{F} = \frac{1}{r} \det \begin{pmatrix} e_r & r e_\theta & e_z \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial z \\ F_r & r F_\theta & F_z \end{pmatrix}$$

Ex:

$$\operatorname{curl} \frac{1}{r} e_\theta = \frac{1}{r} \begin{pmatrix} e_r & r e_\theta & e_z \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial z \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \vec{0}.$$