Lecture 45: Stokes Theorem (§8.3)

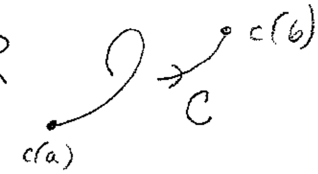
HW: 8.3 # 1,2,3.

Next time: More on 8.3

1-d:

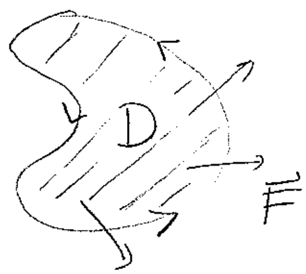
$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

1-d in $\mathbb{R}^3$: $f : \mathbb{R}^3 \to \mathbb{R}$



$$\int_C \nabla f \cdot ds = f(c(b)) - f(c(a))$$

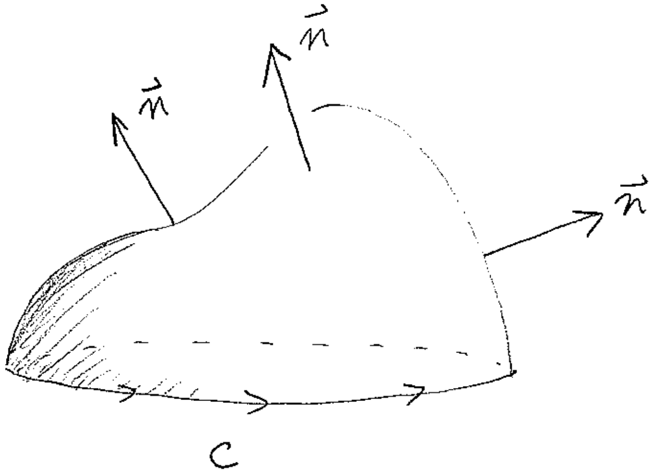
2-d: Green's Thm: D region
$\vec{F}$ a vector field



$$\iint_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\,dA = \int_{\partial D} \vec{F} \cdot ds$$

2-d in $\mathbb{R}^3$:

$$\boxed{\text{Stokes Theorem}}$$

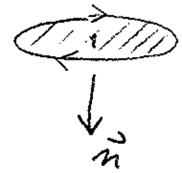


S a surface in $\mathbb{R}^3$ with boundary curve C.
$\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ a vector field
$\vec{n}$ unit normal.

Stokes Theorem: $$\int_C \vec{F} \cdot ds = \iint_S (\text{curl}\,\vec{F}) \cdot \vec{n}\,dA$$

Have to orient C correctly: the surface should be to your left as you walk around C with your head pointing in the normal direction

and



Also, S has to be orientable.

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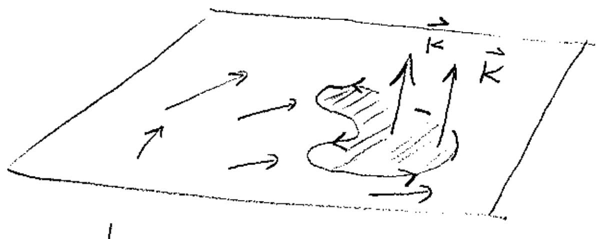
<u>Relation to Green's theorem:</u>



$\vec{F}$ confined in the (x,y)-plane.

$$\text{curl } \vec{F} \cdot \vec{n}$$

$$= (\text{curl } \vec{F}) \cdot k = \text{scalar curl of } \vec{F}$$

$$= \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

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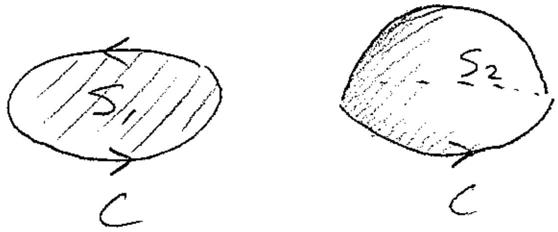
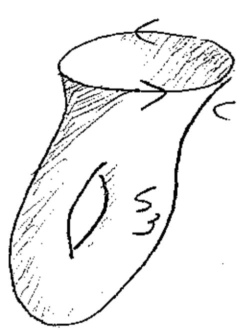
[Stokes Thm is remarkable: not only does it relate the integral over the whole surface with one over the boundary, but the same curve is the boundary of many surfaces!]

$$\iint_{S_1} \text{curl } \vec{F} \cdot dS = \int_C \vec{F} \cdot ds$$

$$\quad\quad \| \quad\quad\quad\quad \|$$



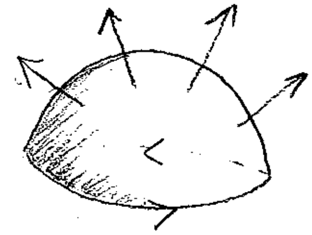


$$\iint_{S_2} \text{curl } \vec{F} \cdot dS \qquad \iint_{S_3} \text{curl } \vec{F} \cdot dS$$

Ex: S_2 = upper unit hemisphere

$\vec{n}$ = outward normal

C = unit circle



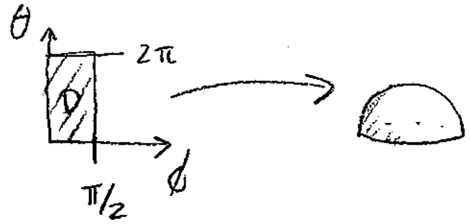
$$\vec{F} = (y, xz, 1) \qquad c(t) = (\cos t, \sin t, 0)$$

$$\int_C \vec{F} \cdot ds = \int_0^{2\pi} (\sin t, 0, 1) \cdot (-\sin t, \cos t, 0)\, dt$$

$$= \int_0^{2\pi} -\sin^2 t \, dt = -\pi$$

$$\text{curl } \vec{F} = \begin{pmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & xz & 1 \end{pmatrix} = (-x, 0, z-1)$$

Param S_2 via r:



$$r(\phi, \theta) = (\sin\phi \cos\theta, \sin\phi \sin\theta, \cos\phi)$$

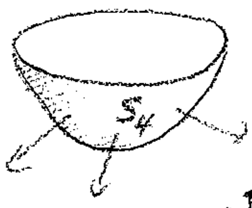
$$\iint_{S_2} (\text{curl } F) \cdot \vec{n} \, dA = \iint_{S_2} (-x, 0, z-1) \cdot (x, y, z)\, dA$$

$$= \iint_{S_2} -x^2 + z^2 - z \, dA = \int_0^{\pi/2} \int_0^{2\pi} (-\sin^2\phi \cos^2\theta + \cos^2\phi - \cos\phi) \sin\phi \, d\theta \, d\phi$$

$$= \int_0^{\pi/2} -\pi \sin^3\phi + 2\pi \cos^2\phi \sin\phi - 2\pi \cos\phi \sin\phi \, d\phi$$

$$= \pi \int_0^{\pi/2} -\sin\phi + 3\cos^2\phi \sin\phi - \cos\phi \sin\phi \, d\phi$$

$$= \pi \left(\cos\phi - \cos^3\phi + \cos^2\phi \right) \Big|_{\phi=0}^{\pi/2} = \boxed{-\pi} \text{ as expected}$$

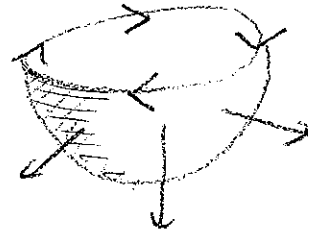
Now consider  the lower hemisphere.

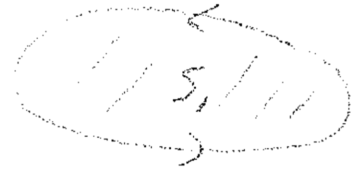
$$\iint_{S_4} (\text{curl } F) \cdot \vec{n} \, dA = \int_{\pi/2}^{\pi} \int_0^{2\pi} (\text{same as before}) \, d\theta \, d\phi$$

$$= \pi \left(\cos\phi - \cos^3\phi + \cos^2\phi \right) \Big|_{\phi=\pi/2}^{\phi=\pi} = \pi$$

<u>Q</u>: What went wrong? <u>A</u>: We used the wrong normal.

Correcting introduces another sign, giving $-\pi$ as before



Also, for 

we have $(\text{curl } F)\cdot\vec{n}$

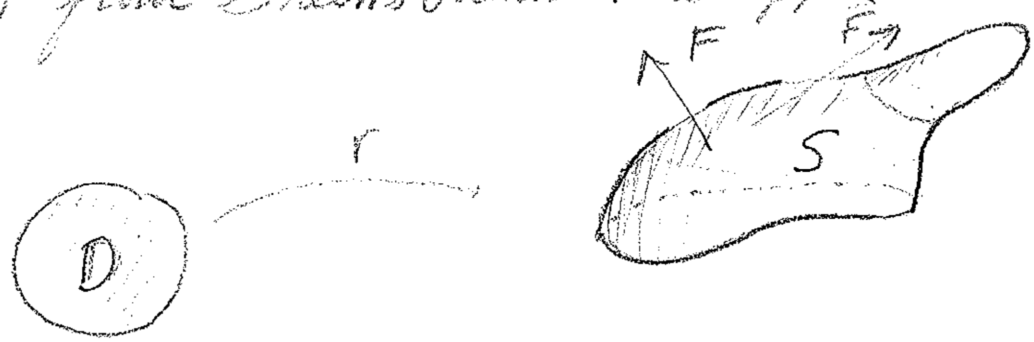
$= (-x, 0, -1)\cdot k = -1$ and so $\iint_{S_1} (\text{curl } \vec{F})\cdot\vec{n}\, dA = (-1)\,\text{Area}(S_1)$

$= \boxed{-\pi}$ ✓

Why is Stokes Theorem true? Like $\int_C \nabla f \cdot ds =$ diff at endpoint,

it is actually an intrinsic thing,

coming from Green's Theorem. Suppose we have



Consider the vector field G on D where

$$\left[Dr(u_0,v_0)\right]\left[G(u_0,v_0)\right] = \text{Proj}_{T_{r(u_0,u_0)}} F(r(u_0,v_0))$$

Then $\begin{array}{c}\text{Scalar curl } G \\ \text{at } u_0,v_0\end{array} = \begin{array}{c}(\text{curl } \vec{F})\cdot\vec{n} \\ \text{at } r(u_0,v_0).\end{array}$