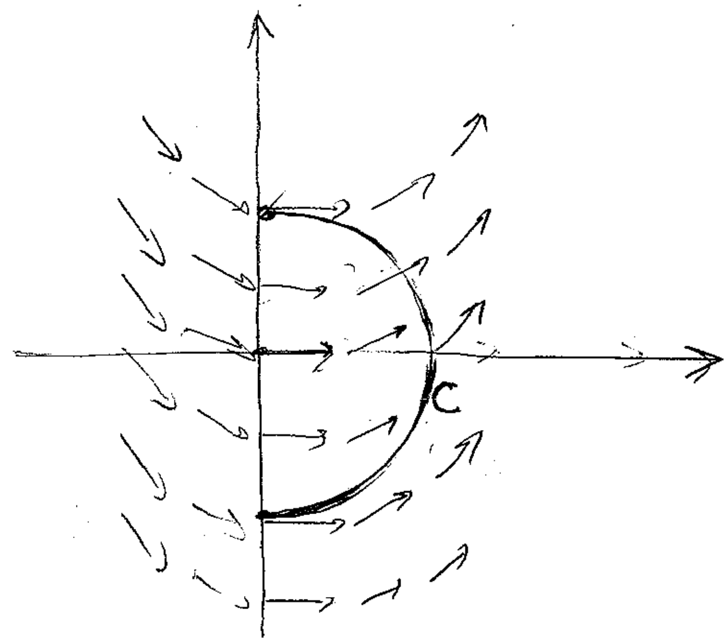
Lecture 39: Integrating vector fields over surfaces (§7.4)

HW: §7.4: 3, 5, 6, 15 (see example 2.90 on pg 146), 21, 23

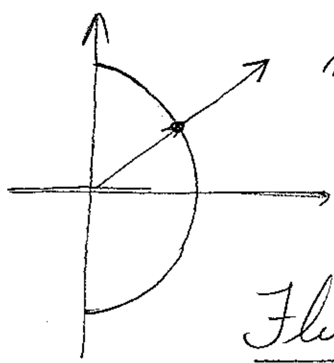
Next time: Starting Chapter 8.

Flux: $\vec{F} = (1, x)$



Parametrize:

$c(t) = (\cos t, \sin t) \quad -\pi/2 \leq t \leq \pi/2$



$n(t) = (\cos t, \sin t)$

Flux: $\displaystyle\int_C (\vec{F} \cdot \vec{n})\, ds = \int_{-\pi/2}^{\pi/2} (F(c(t)) \cdot n(t))\, \|c'(t)\|\, dt$

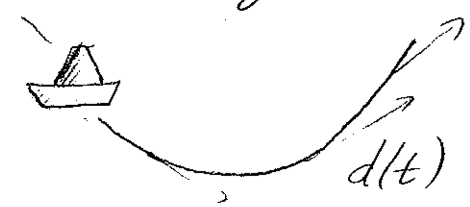
$$= \int_{-\pi/2}^{\pi/2} (1, \cos t) \cdot (\cos t, \sin t)\, dt = \int_{-\pi/2}^{\pi/2} \cos t + \cos t \sin t \, dt$$

$$= 2$$

Recall that the flux represents the amount of water that passes through C in unit time.

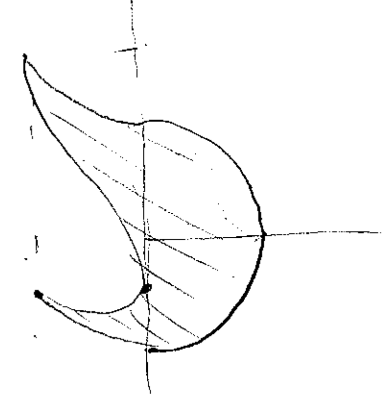
<u>Q</u>: What does this region look like?

First we need to figure out how an object moves in this flow



$$d'(t) = F(d(t))$$

Here $d(t) = (t, \frac{1}{2}t^2)$ works [This is an example of a differential equation...]



S a surface in $\mathbb{R}^3$.

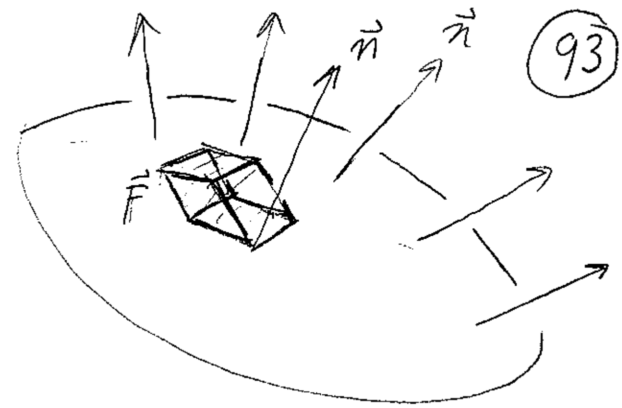
$\vec{F} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ a vector field

The flux is

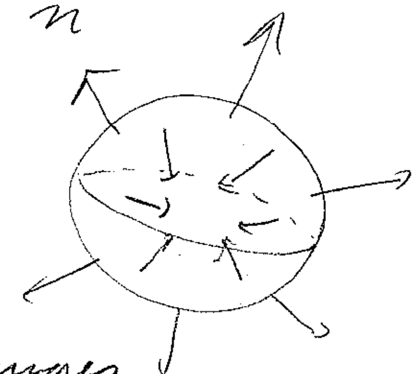
$$\iint_S (\vec{F} \cdot \vec{n})\, dA \overset{\text{notation}}{=} \iint_S \vec{F} \cdot dA$$

where $\vec{n}$ is a unit normal vector field on S.

Again, this measures the rate of flow



Note: There are typically two choices for $\vec{n}$

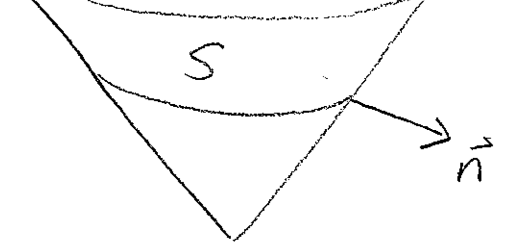


Changing the $\vec{n}$ to $-\vec{n}$ changes the sign of $\iint_S \vec{F} \cdot dA$ [Clear from 1st form.]

Some surfaces have no good choice of normal vector, like Möbius bands  ,
so we can't integrate vector fields over those. [Discuss Klein bottle.]

Ex: Cone $x^2+y^2=z^2$ $\quad 0 \le z \le 1$

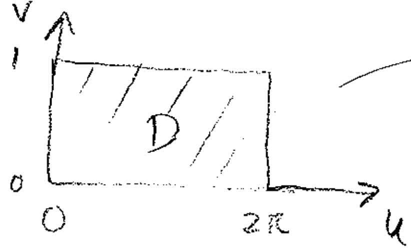
$\vec{F} = (z, 1, x)$



Compute

$$\iint_S \vec{F} \cdot dA$$

Param:



$r(u,v) = (v\cos u, v\sin u, v)$

$$\vec{n} = \frac{T_u \times T_v}{\|T_u \times T_v\|} = \left(\frac{1}{\sqrt{2}}\cos u, \frac{1}{\sqrt{2}}\sin u, -1/\sqrt{2}\right)$$

$$\iint_S \vec{F} \cdot dS = \iint_S (\vec{F} \cdot \vec{n})\, ds =$$

$$\int_0^1 \int_0^{2\pi} \vec{F}(r(u,v)) \cdot \vec{n}(u,v)\, \|T_u \times T_v\|\, du\, dv$$

$$= \int_0^1 \int_0^{2\pi} (v, 1, v\cos u) \cdot \left(\frac{1}{\sqrt{2}}\cos u, 1/\sqrt{2}\sin u, -1/\sqrt{2}\right) \cdot \sqrt{2}\,v\, du\, dv.$$

Note: A shortcut is

$$\iint_S F \cdot dA = \iint_S (F \cdot \vec{n})\, dA = \iint_D \left(F \cdot \frac{T_u \times T_v}{\|T_u \times T_v\|}\right) \|T_u \cdot T_v\|\, du\, dv$$

$$= \iint_D F(r(u,v)) \cdot (T_u \times T_v)\, du\, dv.$$