Lecture 6: Improper Integrals (§6.6)

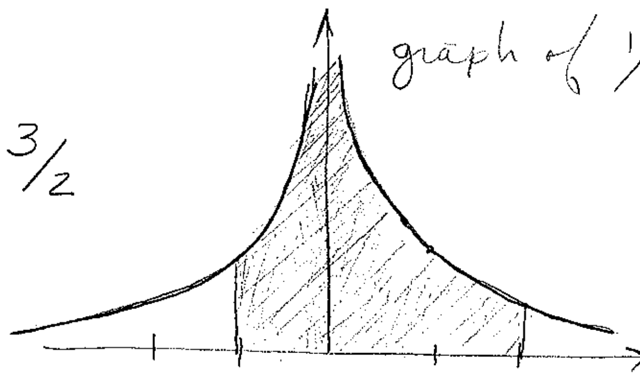
HW #3: Due Sept 10 §6.6 7, 8.

Next time: More §6.6

Improper Integrals: Definite integrals where the Fund Thm of Calc does not apply.

Ex: $\displaystyle\int_{-1}^{2} \frac{1}{x^2}\,dx = \left.\frac{x^{-1}}{-1}\right|_{x=-1}^{x=2} = -3/2$

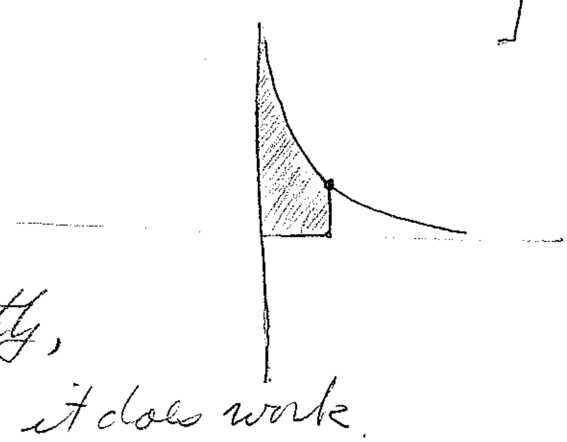
not really



But that can't be right as the graph shows a positive area?! Query: What went wrong? The FToC requires the integrand to be continuous on $[a,b]$ to compute $\int_a^b f(x)\,dx$.

[However, even for discontinuous functions can sometimes be integrated.]

Ex: $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2$

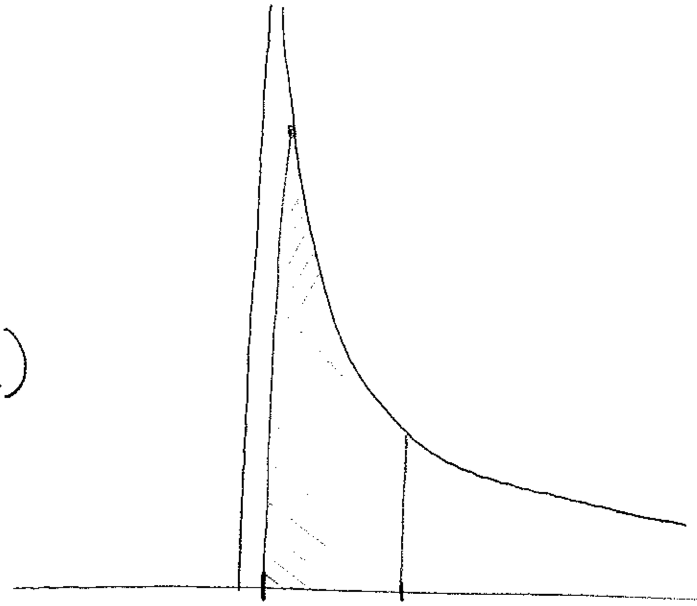


While the FToC doesn't apply directly, can approx by thing where it does work.

For $0 < R < 1$, have

$$\int_R^1 x^{-1/2}\,dx = 2x^{1/2}\Big|_{x=R}^{1}$$

$$= 2 - 2\sqrt{R} = 2(1-\sqrt{R})$$



Eg. If $R = 0.0001$, area is 1.98
$R = 0.000001$, area is 1.998

Indeed

$$\lim_{R\to 0^+} (2 - 2\sqrt{R}) = 2$$ so we say $\int_0^1 x^{-1/2}\,dx = 2$.

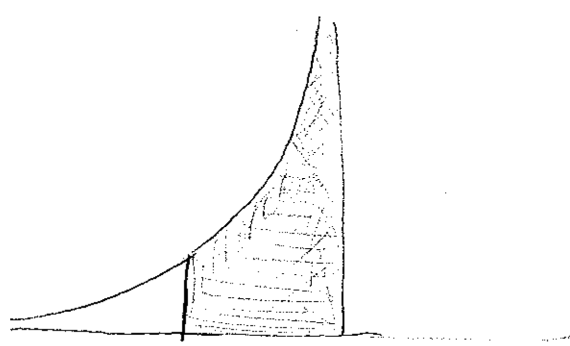
Def: Suppose f is continuous on $(a, b] = \{a < x \leq x\}$

Then we set $$\int_a^b f(x)\,dx = \lim_{R\to a^+} \int_R^b f(x)\,dx$$ talk about what this means.

provided the limit exists, in which case we say the improper integral converges. [Otherwise it diverges]

Similar if f is cont on $[a, b)$.

Ex: $$\int_{-1}^0 x^{-2}\,dx = \lim_{R\to 0^-} \int_{-1}^R x^{-2}\,dx$$

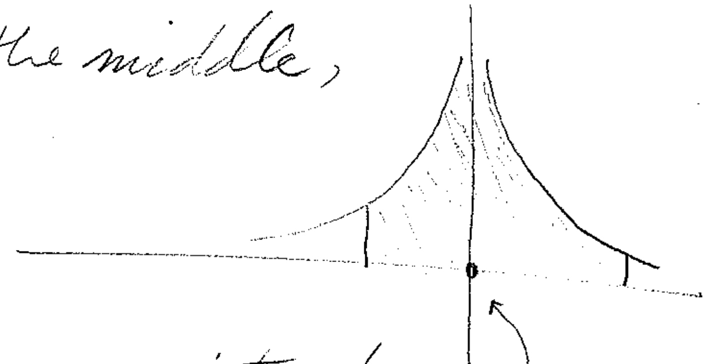


$$\lim_{R\to 0^-} -x^{-1}\Big|_{-1}^{R} = \lim_{R\to 0^-} (1 - 1/R)$$ which does not exist.

So this integral diverges.

__Note:__ It is reasonable to say that there is infinite area here

What about discontinuities in the middle, as in $\int_{-1}^{2} x^{-2}\,dx$?

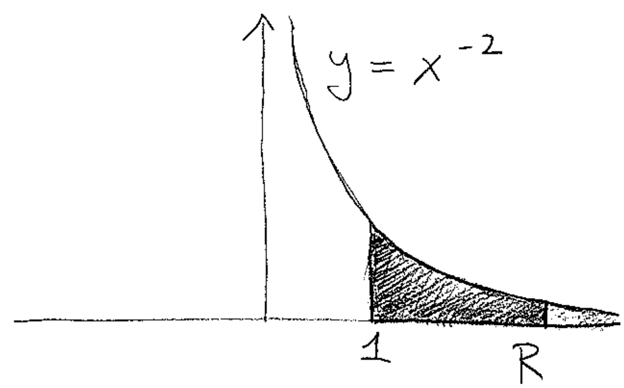


Can break up into simpler improper integrals.

$$\int_{-1}^{2} x^{-2}\,dx = \int_{-1}^{0} x^{-2}\,dx + \int_{0}^{2} x^{-2}\,dx$$

where the integral converges only if __both__ new integrals do. In this case, the integral diverges.

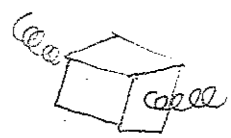
__Infinite limits of integration__



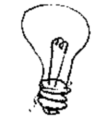
$$\int_{1}^{\infty} x^{-2}\,dx = \lim_{R\to\infty} \int_{1}^{R} x^{-2}\,dx$$

$$= \lim_{R\to\infty} -x^{-1}\Big|_{1}^{R} = \lim_{R\to\infty} -\frac{1}{R} + 1 = 1.$$

[Talk about why such things are useful.]

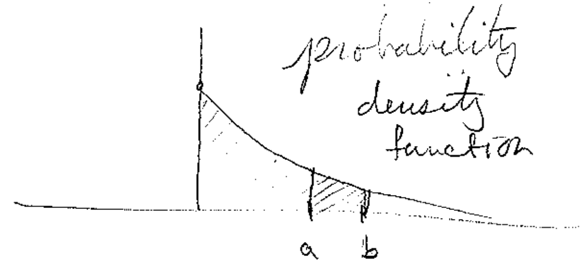
<u>Component failure:</u> 

<u>Model:</u> probability of failure is <u>independent</u> of age.

[e.g. ]

$\rightsquigarrow$ prob of failing during time $[a,b]$ given by $\int_a^b ce^{-cx}\,dx$

where c = rate of failure



<u>Note</u>

$$\int_0^\infty c e^{-cx}\,dx = \lim_{R\to\infty} \int_0^R ce^{-cx}\,dx$$

$u = -cx$
$du = -c\,dx$

$$= \lim_{R\to\infty} \int_0^{-cR} -e^u\,du$$

$$= \lim_{R\to\infty} -e^{-cR} + e^0 = 1$$

That is everything fails eventually.

<u>Mean Time to Failure:</u>

$$\int_0^\infty x c e^{-cx}\,dx =$$