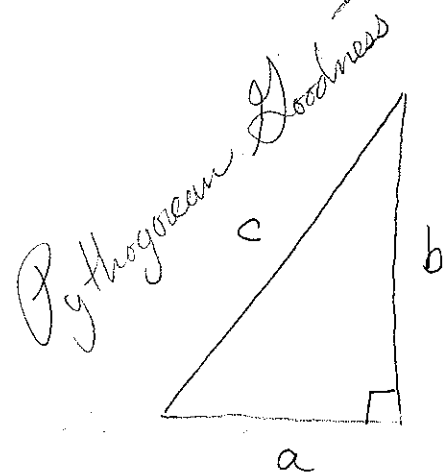
Lecture 5: Trig substitution (§6.3)

HW #3 (Due Sept 10) 6.3 # 15, 16, 23, 26
others.

Next time: §6.6 (Will come back to 6.4-6.5 later.)

Comment on: showing work on HW; honors problem set 1.

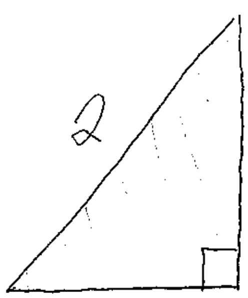
Trig substitution:



$a^2 + b^2 = c^2$
Or equivalently,
$a^2 = c^2 - b^2$

Ex: $\int \frac{1}{x^2 \underbrace{\sqrt{4-x^2}}} \, dx$

Goal: get rid of this, i.e. make $4-x^2$ a square.

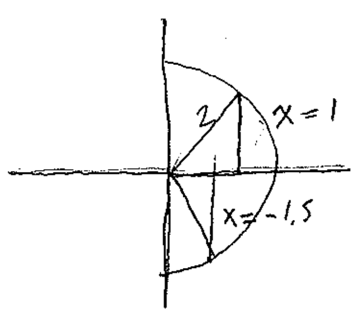


$x = 2\sin\theta$

$\sqrt{4-x^2} = 2\cos\theta$

Note: For $\sqrt{4-x^2}$ to make sense, need $4-x^2 \geq 0$, that is, $-2 \leq x \leq 2$. So $-\pi/2 \leq \theta \leq \pi/2$

Now, as $dx = 2\cos\theta \, d\theta$



$$\int \frac{1}{x^2\sqrt{4-x^2}} dx = \int \frac{2\cos\theta}{(2\sin\theta)^2 \, 2\cos\theta} d\theta$$

$$= \frac{1}{4} \int \frac{1}{\sin^2\theta} d\theta = \frac{1}{4} \int \csc^2\theta \, d\theta$$

$$= -\frac{1}{4}\cot\theta + C$$

But what is $\uparrow$ in terms of x? Well

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{2\cos\theta}{2\sin\theta} = \frac{\sqrt{4-x^2}}{x}$$

So $\displaystyle\int \frac{1}{x^2\sqrt{4-x^2}}\,dx = -\frac{1}{4}\,\frac{\sqrt{4-x^2}}{x} + C$

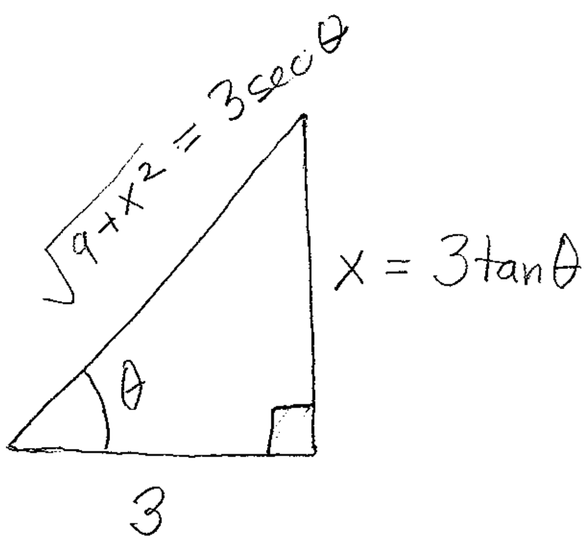
<u>Moral</u>: When you see $\sqrt{a^2-x^2}$, try $x = a\sin\theta$.

Have similar methods for $\underbrace{\sqrt{a^2+x^2}}_{x = a\tan\theta}$, $\underbrace{\sqrt{x^2-a^2}}_{x = a\sec\theta}$

<u>Ex</u>: $\displaystyle\int \frac{1}{\sqrt{9+x^2}}\,dx$

$\sin\theta = \frac{o}{h}$ $\quad\cos\theta = \frac{a}{h}$

$\tan\theta = \frac{o}{a}$



Then

$$\sqrt{9+x^2} = a\cdot\left(\frac{h}{a}\right)$$

$$= 3\cdot\sec\theta$$

Check algebraically $\qquad \sec^2\theta - \tan^2\theta = 1$

$$\sqrt{9 + (3\tan^2\theta)} = \sqrt{9 + 9\tan^2\theta} = 3\sqrt{1+\tan^2\theta}$$

$$= 3\sqrt{\sec^2\theta} = 3\sec\theta$$

as implicitly we are using $-\pi/2 < \theta < \pi/2$ where $\cos\theta \geq 0$. Either way

$$\int \frac{1}{\sqrt{9+x^2}}\,dx = \int \frac{1}{3\sec\theta}\cdot \underbrace{3\sec^2\theta\, d\theta}_{= dx}$$

$$= \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C$$

will explain in a minute

Now $\tan\theta = \dfrac{x}{3}$, $\sec\theta = \frac{1}{3}\sqrt{9+x^2}$,

so
$$\int \frac{1}{\sqrt{9+x^2}}\,dx = \ln\left|\frac{1}{3}\sqrt{9+x^2} + \frac{x}{3}\right| + C$$

$$= \ln \underbrace{\frac{1}{3}\left(\sqrt{9+x^2} + x\right)}_{\text{always positive}} + C$$

[For remaining type $\sqrt{x^2-a^2}$ see the text]

1

Remaining kinds of trig integrals

$$\int \tan^n x \sec^n x \, dx$$

Typically, do a substitition of $u = \sec x$
$du = \sec x \tan x$

$$\int \tan^3 x \sec^3 x \, dx = \int \tan^2 x \sec^2 x (\sec x \tan x) \, dx$$

$$= \int (\sec^2 x - 1) \sec^2 x (\sec x \tan x) \, dx$$

$$= \int (u^2 - 1) u^2 \, du = \ldots$$

Similar for other sorts of trig functions.

Key example: $\int \sec x \, dx = \int \sec x \, \frac{\sec x + \tan x}{\sec x + \tan x} \, dx$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx = \int \frac{1}{u} \, du$$

$u = \sec x + \tan x$

$$= \ln | \sec x + \tan x | + C$$