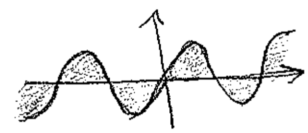
Lecture 4: Integrating Trig Functions (§6.3)

HW #3 (Due Sept 10) 6.2 40, 59, 60
6.3 5, 8, 17, 20

Next time: Rest of §6.3

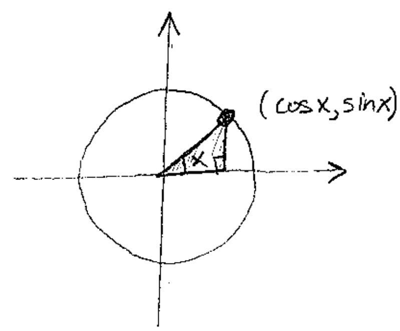
Comment on: HW policy; things you don't know
high school vs. college math.



Methods for: $\int \sin^m x \cos^n x \, dx$ for n, m positive integers.
[Motivate: waves, trig situation.]

Substitution: $\int \sin^2 x \cos x \, dx = \int u^2 \, du = \frac{1}{3}u^3 + C$

$u = \sin x$
$du = \cos x \, dx$
$= \frac{1}{3}\sin^3 x + C$



Identities: $\sin^2 x + \cos^2 x = 1$

$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \quad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$

[Come from $\cos 2x = \cos^2 x - \sin^2 x$]

Ex: $\int \sin^2 x \cos^3 x \, dx$ [Goal: $\int \left(\begin{array}{c}\text{something} \\ \text{only involving} \\ \sin\end{array}\right) \cos x \, dx$]

$= \int \sin^2 x \cos^2 x \cos x \, dx = \int \sin^2 x (1 - \sin^2 x) \cos x \, dx$

$u = \sin x \quad du = \cos x \, dx$

$= \int u^2(1-u^2) \, du = \int u^2 - u^4 \, du = \frac{u^3}{3} - \frac{u^5}{5} + C = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$

The method we used works as long as one of n or m is odd. Eg.

$$\int \underbrace{\sin^5 x}_{= (1-\cos^2 x)^2 \sin x} \cos^4 x \, dx = \int (1-\cos^2 x)^2 \cos^4 x \sin x \, dx$$

$u = \cos x$
$du = -\sin x \, dx$

What about when they are both even?

Ex: $\int \sin^2 x \, dx = \int \frac{1}{2}(1-\cos 2x)\, dx = \frac{1}{4}\int (1+\cos 2x)\, 2\, dx$

$u = 2x$
$du = 2x$

$$= \frac{1}{4}\int 1 + \cos u \, du = \frac{1}{4}(u + \sin u) + C$$

$$= \frac{1}{4}(2x + \sin 2x) + C.$$

Ex: $\int \sin^2 x \cos^2 x \, dx = \int \sin^2 x (1-\sin^2 x)\, dx$

$$= \int \sin^2 x \, dx - \int \sin^4 dx = \underbrace{\int \sin^2 x \, dx}_{\text{just did}} - \underbrace{\int \left(\tfrac{1}{2}(1-\cos 2x)\right)^2 dx}_{-\frac{1}{4}\int 1 - 2\cos 2x + \underbrace{\cos^2 2x}_{\text{made prodgess}} \, dx}$$

Similar procedure applies to other kinds of simple trig integrals, like

$\int \tan^n x \sec^m x \, dx$ [More on this next time.]

<u>Trig Substitution:</u> Things like $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, or $\sqrt{x^2-a^2}$

<u>Recall:</u> $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$

<u>Ex:</u> $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \int \frac{\cos\theta}{\underbrace{\sqrt{1-\sin^2\theta}}_{=\cos^2\theta}}\,d\theta = \int d\theta = \theta + C$

$= \sin^{-1}x + C$

$x = \sin\theta$ (or $\theta = \sin^{-1}x$)

$dx = \cos\theta\, d\theta$

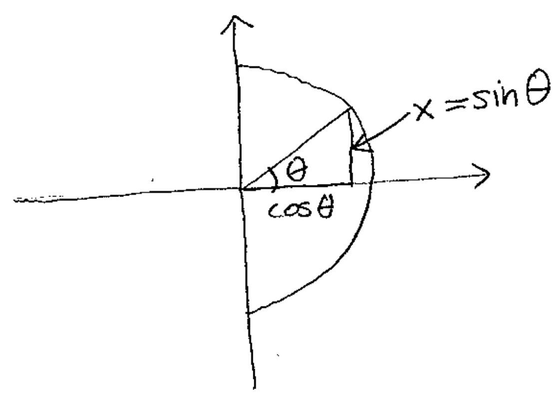
More carefully, for $\dfrac{1}{\sqrt{1-x^2}}$ to make sense,

we must have $-1 < x < 1$

Our new variable θ will sat $-\pi/2 < \theta < \pi/2$

so that $\cos\theta$ is always positive,

and so $\sqrt{\cos^2\theta} = \cos\theta$.



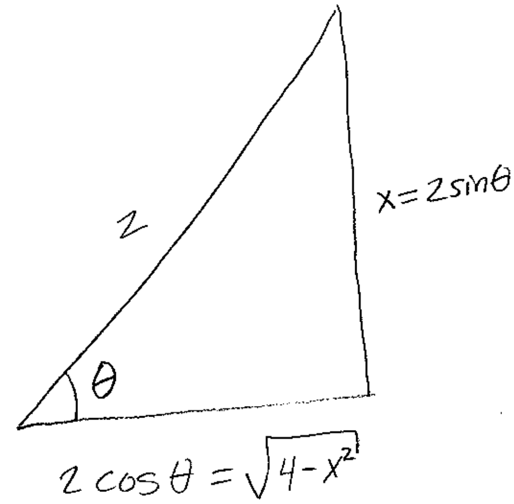
<u>Ex:</u> $\displaystyle\int \frac{1}{x^2\sqrt{4-x^2}}\,dx$

$x = 2\sin\theta$

$dx = 2\cos\theta\, d\theta$

$\sqrt{4-x^2} = \sqrt{4-(2\sin\theta)^2} = \sqrt{4(1-\sin^2\theta)} = 2\cos\theta$

$$\int \frac{1}{x^2\sqrt{4-x^2}}\,dx = \int \frac{2\cos\theta}{(2\sin\theta)^2 2\cos\theta}\,d\theta$$



$$= \frac{1}{4}\int \frac{1}{\sin^2\theta}\,d\theta = \frac{1}{4}\int \csc^2\theta\,d\theta$$

$$= -\frac{1}{4}\cot\theta + C$$

But what is $\cot\theta$?! Well

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\sqrt{4-x^2}}{x}$$

So

$$\int \frac{1}{x^2\sqrt{4-x^2}}\,dx = -\frac{1}{4}\,\frac{\sqrt{4-x^2}}{x} + C.$$