Lecture 14:

HW # 5: Due Oct 1 No additional problems.

Next time: §8.2

Monotone Sequences:

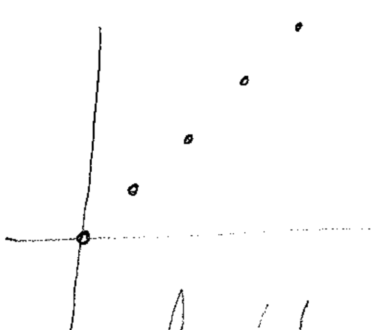
Increasing: $a_1 \le a_2 \le a_3 \le a_4 \le a_5 \le \dots$

Decreasing: $a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5 \geq \dots$

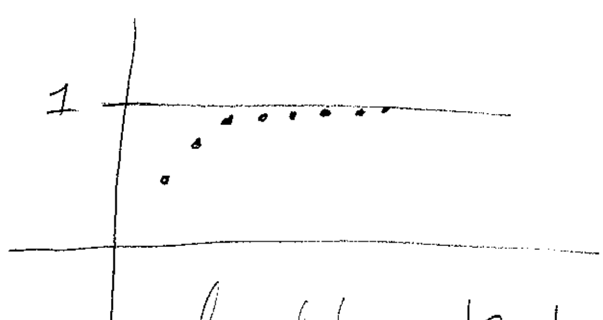
A sequence is bounded if there is an M so that $|a_n| \le M$ for all n.

Thm: A monotone sequence converges if and only if it is bounded.

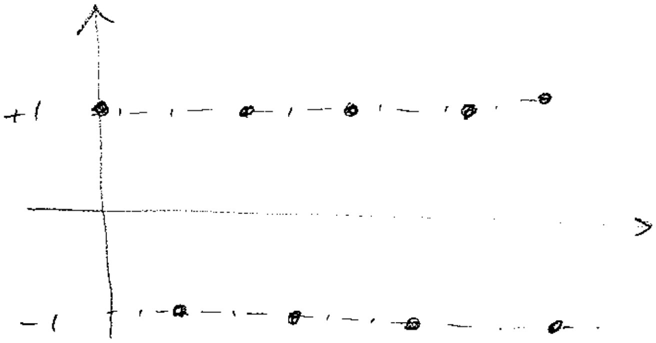
Ex: $\{a_n = n\}_{n=0}^{\infty}$ $\qquad$ $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$



unbounded, divergent

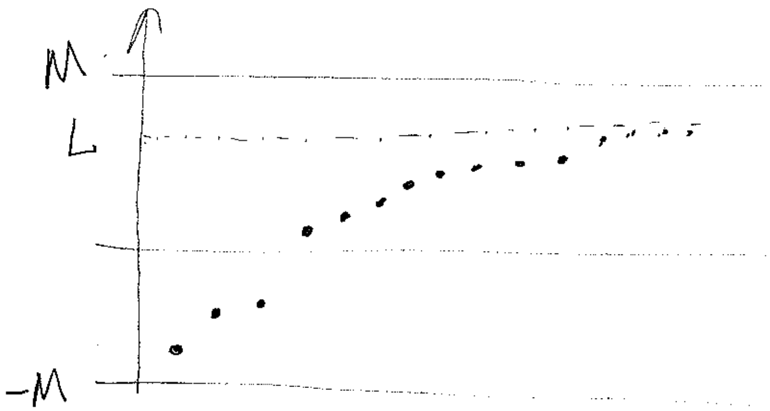


bounded: $|a_n| \le 1$, convergent.



non-monotone, bounded, but divergent

<u>Idea</u>: Suppose $\{a_n\}$ is an increasing sequence with $|a_n| \leq M$, and thus $-M \leq a_n \leq M$.



If we take L to be the <u>smallest</u> number where $a_n \leq L$ for all n, then

$$\lim_{n\to\infty} a_n = L$$

[That there is such a smallest upper bound is the <u>Completeness Property of The Real Numbers</u>]

<u>Ex</u>: $$\left\{a_n = \frac{2^n}{n!}\right\}_{n=1}^{\infty} = \left\{2, 2, 4/3, 4/15, 4/45, \ldots\right\}$$

A decreasing sequence, since $\dfrac{a_{n+1}}{a_n} = \dfrac{2}{n+1} \leq 1$

Also bounded: $0 \le a_n \le a_1 = 2$, so $|a_n| \le 2$.

Hence, it converges. What's the limit? $a_{12} = 0.0000086$, $a_{20} = 4.31 \times 10^{-13}$ suggests 0.

Indeed,

$$a_n = \frac{2^n}{n!} = \frac{2 \cdot \overbrace{2 \cdot 2 \cdot 2 \cdots 2}^{n}}{1 \cdot \underbrace{2 \cdot 3 \cdot 4 \cdots n}_{\le 1}} \le \frac{4}{n}$$

So

$$0 \le a_n \le 4/n \overset{\text{Squeeze Thm}}{\Longrightarrow} \underbrace{\lim_{n\to\infty} 0}_{=0} \le \lim_{n\to\infty} a_n \le \underbrace{\lim_{n\to\infty} 4/n}_{=0}$$

and so $\lim_{n\to\infty} a_n = 0$.

Infinite Series (Sums)

$$1/2 + 1/4 + 1/8 + 1/16 + \cdots = 1$$
$$1 + 1/4 + 1/9 + 1/16 + 1/25 + \cdots = \pi^2/6$$

What does this mean? We only know how to add up finitely many things. [Remind of Σ notation]

$$\sum_{k=1}^{5} 1/2^k = 1/2 + 1/4 + 1/8 + 1/16 + 1/32 = 31/32 \quad \text{and}$$

$$\sum_{k=1}^{10} 1/2^k = 1/2 + 1/4 + 1/8 + \cdots + 1/256 + 1/512 + \frac{1}{1024} = \frac{1023}{1024}$$

$$S_1 = 1/2$$

$$S_2 = 1/2 + 1/4 = 3/4$$

$$S_3 = 1/2 + 1/4 + 1/8 = \frac{7}{8}$$

$$S_4 = 1/2 + 1/4 + 1/8 + 1/16 = 15/16$$

Partial sums

$$\vdots$$

$$S_n = 1/2 + 1/4 + 1/8 + \cdots + 1/2^k = \sum_{k=1}^{n} 1/2^k = \frac{2^n - 1}{2^n} = 1 - 1/2^n$$

Now $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} 1 - 1/2^n = 1$ so we say

$$\sum_{k=1}^{\infty} 1/2^k = 1.$$

In general, suppose $\{a_k\}_{k=1}^{\infty}$ is a sequence and we want to define $\sum\limits_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + a_4 + \cdots$

Let

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$

If $\lim\limits_{n\to\infty} S_n$ exists and is equal to S, we say that $\sum\limits_{k=1}^{\infty} a_k$ converges and write $\sum\limits_{k=1}^{\infty} a_k = S$.