Lecture 13: Sequences, part II (§8.1)

HW#5: Due Oct 1 : §8.1 # 21, 24, 33, 36, 40, 41

Next time: More on §8.1, start §8.2

Last time: Sequences: $\{1/n\}_{n=1}^{\infty} = \{1, 1/2, 1/3, 1/4, 1/5, \ldots\}$

Def: A sequence $\{a_n\}_{n=n_0}^{\infty}$ converges to L (i.e. $\lim_{n\to\infty} a_n = L$) if for every $\varepsilon > 0$ there is an N so that

$$|a_n - L| < \varepsilon \quad \text{for each } n \geq N.$$

$$a_1, a_2, a_3, a_4, a_5, a_6, \underbrace{a_7, a_8, \ldots}_{\text{all within } \varepsilon \text{ of } L.}$$

ε is for "error": [Initially, we will focus on qualitative statements, i.e. does the limit exist. However quantitative things are also important, e.g. how quickly does a_n approach L?]

$$\lim_{n\to\infty} (1 + 1/n)^n = e = \underbrace{2.718}_{\text{just want to find initial part}}28182 84\ldots$$

That is want n so that $a_n = (1+1/n)^n$ is close to e, say precisely that the error $|a_n - e| < \underbrace{0.0001}_{\text{max allowed error}}$.

Turns out, need $n = 15,000$ or so:

$$\left(1+\frac{1}{15000}\right)^{15000} = \left(\frac{15001}{15000}\right)^{15000} = 2.718192246$$

<u>Rules for limits:</u> Suppose $\{a_n\}_{n=n_0}^{\infty}$ and $\{b_n\}_{n=n_0}^{\infty}$ both converge.

Then i) $\lim\limits_{n\to\infty} a_n + b_n = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n$ (also $a_n - b_n$)

ii) $\lim\limits_{n\to\infty} a_n b_n = \left(\lim\limits_{n\to\infty} a_n\right)\left(\lim\limits_{n\to\infty} b_n\right)$

iii) $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$ provided $\lim\limits_{n\to\infty} b_n \neq 0$

<u>Relation to limits of functions:</u>

<u>Thm:</u> If $\lim\limits_{x\to\infty} f(x) = L$ then $\lim\limits_{n\to\infty} f(n) = L$

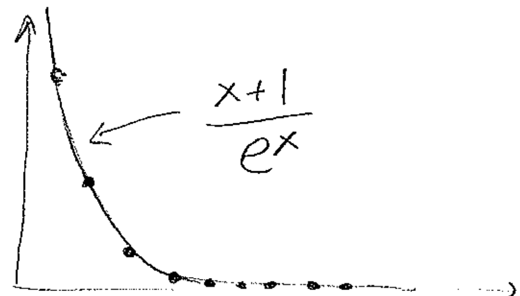
<u>Ex:</u> $\lim\limits_{x\to\infty} \sin(1/x) = 0$ so therefore $\lim\limits_{n\to\infty} \sin(1/n) = 0$.

Tricky bit: $\lim\limits_{n\to\infty} f(n)$ can exist even if $\lim\limits_{x\to\infty} f(x)$ does not.

Ex: $\lim\limits_{n\to\infty} \cos 2\pi n = 1$ as $\{\cos 2\pi n\}_{n=1}^{\infty} = \{1, 1, 1, 1, \dots\}$

but $\lim\limits_{n\to\infty} \cos 2\pi x$ does not exist.

Ex: $\lim\limits_{n\to\infty} \dfrac{n+1}{e^n} = 0$ because



L'Hopital for $\frac{\infty}{\infty}$

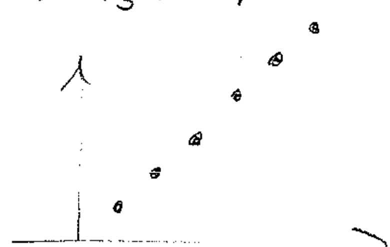
$$\lim_{x\to\infty} \frac{x+1}{e^x} = \lim_{x\to\infty} \frac{\frac{d}{dx}(x+1)}{\frac{d}{dx}(e^x)} = \lim_{x\to\infty} \frac{1}{e^x} = 0$$

Monotone Sequences:

$\{a_n\}_{n=1}^{\infty}$ is <u>increasing</u> if $a_1 \le a_2 \le a_3 \le a_4 \le a_5 \le \cdots$

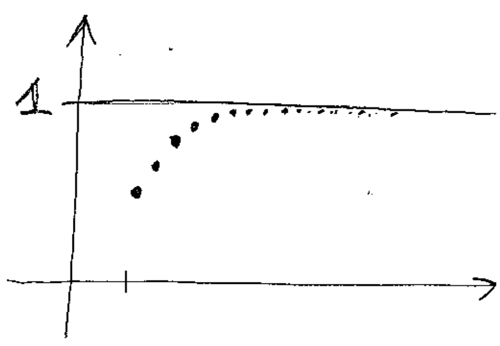
is <u>decreasing</u> if $a_1 \ge a_2 \ge a_3 \ge a_4 \ge \cdots$

Ex of increasing sequences: $\{a_n = n\}$



$\left\{a_n = \dfrac{n}{n+1}\right\}$

$= \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6}, \frac{6}{7}, \dots\right\}$



Check that a_n is increasing; equivalently $a_{n+1} - a_n \geqslant 0$ for every n.

$$a_n = \frac{n}{n+1} \qquad a_{n+1} = \frac{n+1}{n+2} \qquad \left(\text{or } \frac{a_{n+1}}{a_n} \geqslant 1\right)$$

Now

$$a_{n+1} - a_n = \frac{n+1}{n+2} - \frac{n}{n+1} = \frac{(n+1)^2 - n(n+2)}{(n+2)(n+1)}$$

$$= \frac{n^2+2n+1 - n^2 - 2n}{(n+2)(n+1)} = \frac{1}{(n+2)(n+1)} > 0.$$

So $\{a_n\}$ is indeed increasing.

Now $\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{n(1/n)}{(n+1)(1/n)} = \lim_{n\to\infty} \frac{1}{1+1/n} = 1$

Not every increasing sequence converges, e.g $\{n\}_{n=1}^{\infty}$.

A sequence is <u>bounded</u> if there is an $M > 0$ so that

$|a_n| \leq M$ for all M

<u>Bounded</u>: $\left|\frac{n}{n+1}\right| \leq 1$ <u>Unbounded</u>: $\{n\}_{n=1}^{\infty}$

<u>Key</u>: A bounded monotone sequence converges.

Ex: $n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot n$

$1! = 1 \quad 2! = 2 \quad 3! = 6 \quad 4! = 24 \quad 10! = 3{,}628{,}800$

Look at $\left\{ \dfrac{2^n}{n!} \right\}_{n=1}^{\infty} = \left\{ 2, 2, 4/3, 2/3, 4/15, 4/45, \ldots \right\}$

Note: Can't use L'Hopital.

This is decreasing:

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{2^{n+1}}{(n+1)!}}{\dfrac{2^n}{n!}} = \frac{2^{n+1}}{(n+1)!} \frac{n!}{2^n} = \frac{2}{n+1} \leq 1 \text{ for } n \geq 1.$$

and bounded $0 \leq a_n \leq a_1 = 2$, i.e. $|a_n| \leq 2$.

So it converges.