Lecture 12: Sequences (§8.1)

HW #5 Due Wed Oct 1: §7.1 5, 15, 21, 32

Turn in Honor's Set. §8.1 4, 11, 13, 45

Next time: More on §8.1

Next week: Infinite Series: $1 + 1/4 + 1/9 + 1/16 + 1/25 + \cdots = \frac{\pi^2}{6}$

Sequences: An infinite list of numbers

first term

$\{1, 1/2, 1/3, 1/4, 1/5, 1/6, 1/7, \ldots, 1/n, \ldots\}$

second term

$\{1, -1, 1, -1, 1, -1, 1, -1, \ldots\}$

$\{1, 3, 5, 7, 9, 11, 13, \ldots\}$

$\{3, 1, 4, 1, 5, 9, 2, 6, 5, 3, 5, 8, 9, \ldots\}$

Individual numbers are called terms, often denoted $\{a_1, a_2, a_3, a_4, \ldots\}$. Usually specify by a rule: $a_n = \frac{1}{n}$ for $n = 1, 2, 3, \ldots$

Write $\{a_n\}_{n=1}^{\infty} = \{1/n\}_{n=1}^{\infty}$.

$a_1 = 1 \quad a_3 = 1/3$
$a_2 = 1/2 \quad a_4 = 1/4$

Examples: $\{-1^n\}_{n=0}^{\infty}$, $\{2n+1\}_{n=0}^{\infty}$, $\{n^{th}\text{ digit of }\pi\}_{n=1}^{\infty}$

$\{A_n\}$ from honors set.

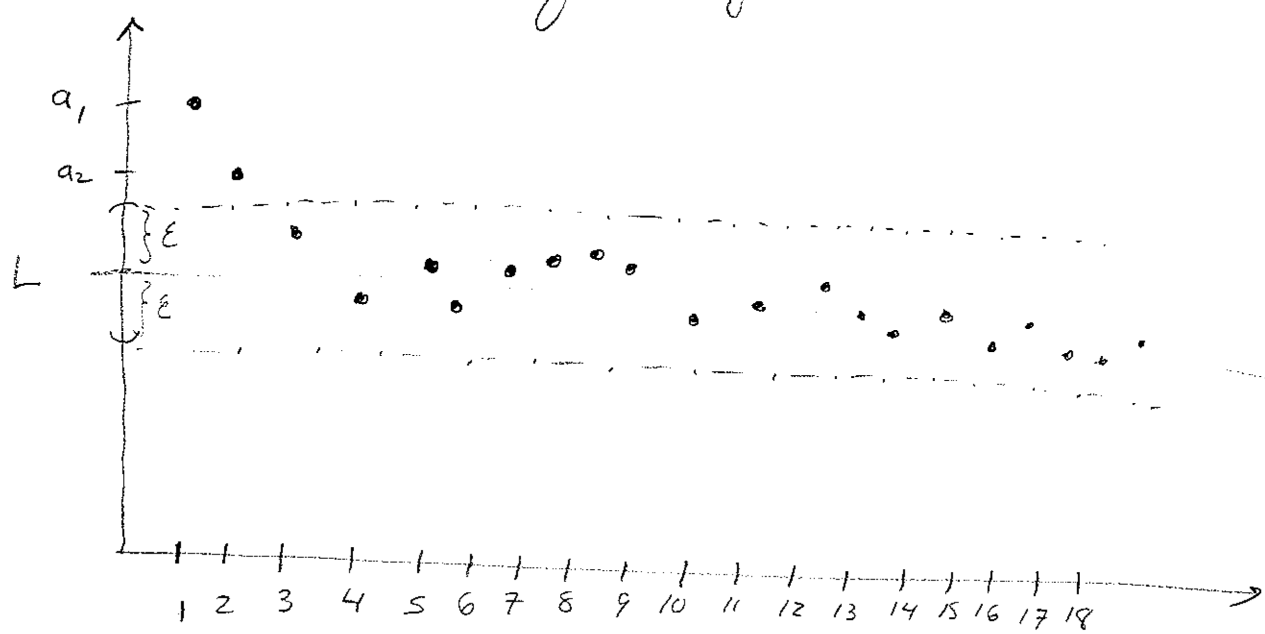
Limits: $a_n = 1/n$ for $n = 1, 2, 3, \ldots$: $1, 1/2, 1/3, 1/4, 1/5, 1/6, \ldots$

Clearly $\lim_{n \to \infty} a_n = 0$, but what does this mean precisely?

Idea: a_n is very close to 0 if n is large enough

Def: A sequence $\{a_n\}_{n=n_0}^{\infty}$ converges to L (i.e. $\lim_{n\to\infty} a_n = L$) if for every $\varepsilon > 0$ there is an N so that

$$|a_n - L| < \varepsilon \quad \text{for every } n > N.$$



ε as shown, take $N = 3$

<u>Ex</u>: $\left\{a_n = 1/n^2\right\}_{n=1}^{\infty} = \left\{1, 1/4, 1/9, 1/16, 1/25, \dots\right\}$

<u>Claim</u>: $\lim\limits_{n\to\infty} 1/n^2 = 0$

<u>Warm Up</u>: Challenge / Response

Given $\varepsilon = 1/5$, I'll take $N=4$. Now for $n \geq N = 4$ we have

$$|a_n - L| = |1/n^2| \leq 1/16 < 1/5 = \varepsilon \quad ✓$$

[Repeat a couple times]

<u>Proof</u>: Let $\varepsilon > 0$ be given. Choose an integer N where $N > \sqrt{1/\varepsilon}$. Then for $n \geq N$ we have

$$|a_n - L| = |1/n^2 - 0| = 1/n^2 \leq \frac{1}{N^2} < \varepsilon$$

since $N > \sqrt{1/\varepsilon} \Rightarrow N^2 > 1/\varepsilon \Rightarrow \varepsilon > \frac{1}{N^2}$.

<u>Can also think about numerically</u>:

	a_n
1	1.0000000
2	0.2500000
3	0.1111111
4	0.0625000
5	0.0400000
6	0.0277777
⋮	
10	0.0100000
100	0.000100

In fact, the notion of limit is (roughly) equivalent to say that for each n, the first n digits of the numbers in the table stop changing after some point. <u>Ex</u>: $a_n = \cos^{-1}(-1 + 1/e^{3n})$

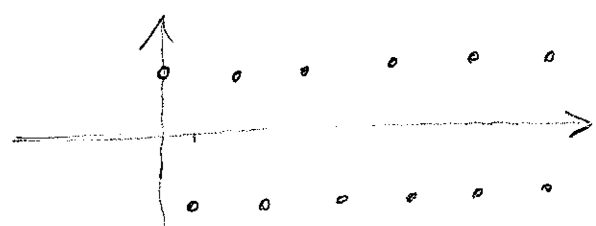
<u>Warning</u>: Just looking at a finite table of numbers is not a proof. But it is good evidence...

<u>Divergent Examples:</u>

1) $$\lim_{n \to \infty} \frac{n^2+1}{2n+3} = \lim_{n \to \infty} \frac{n^2+1}{2n+3} \frac{(1/n)}{(1/n)} = \lim_{n \to \infty} \frac{n + 1/n}{2 + 3/n} = \infty$$

So $\left\{\frac{n^2+1}{2n+3}\right\}_{n=1}^{\infty}$ diverges

2) $\{1, -1, 1, -1, 1, -1, 1, -1, \dots\}$ also diverges.



<u>Proof</u>: (only if asked)

Suppose $\{a_n = (-1)^n\}$ converges to L

Then for $\varepsilon = 1/2$, there is a N s.t. $|a_n - L| < 1/2$ for all $n \geq N$

But $a_N, a_{N+1} = 1, -1$ in some order, and no L is $1/2$ from both of these, a contradiction.