Lecture 11: Differential Equations (§7.1)

HW #5: Due Oct 1. §7.1: #

Next time: 8.1.

Simple Example:

$p(t)$

$$P'(t) = 10 \quad \Rightarrow \quad \int p'(t)\,dt = \int 10\,dt \quad \text{and so}$$

rate of change

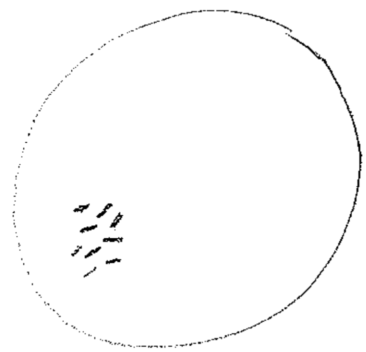
$$p(t) + C_1 = 10t + C_2 \Rightarrow p(t) = 10t + C$$

If car starts off at home which is at $x = 20$, then

$$p(t) = 10t + 20$$

Exponential Growth:

$p(t)$ = population of bacteria at time t



$p(t) = ?$

$$p'(t) = \overset{\text{reproduction rate}}{k}\ \underbrace{p(t)}_{\text{current pop}}$$

Then

$$\frac{p'(t)}{p(t)} = k \quad \text{and so} \quad \int \frac{p'(t)}{p(t)}\,dt = \int k\,dt = kt + C$$

$$= \int \frac{1}{u}\,du \quad \text{with } u = p(t)$$

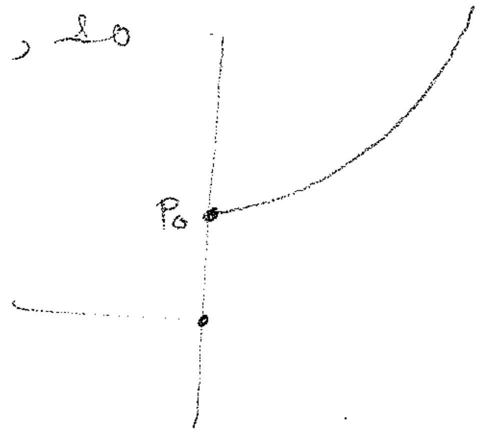
$$= \ln|p(t)|$$

So $\ln p(t) = kt + C$ or

$$p(t) = e^{kt+c} = e^c e^{kt} = A e^{kt}$$

Plugging in $t=0$ gives: $p(0) = A$, so

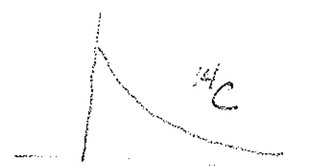
$$p(t) = \underbrace{P_0}_{\text{initial pop}} e^{kt}$$



Time needed to double the population depends only on k, not p_0: $p(t) = 2p_0 \Rightarrow 2p_0 = p_0 e^{kt} \Rightarrow t = \frac{1}{k}\ln 2$.

[Assuming growth is unconstrained.]

Other examples: Radioactive decay (k negative)
bank interest / depreciation



Newton's Law of Cooling: $y(t)$ = temperature at time t

$$y'(t) = \overset{\text{constant} < 0}{k}\,\underbrace{(y(t) - T_a)}_{\text{temperature differential},\ > 0.}$$

T_a = ambient temp of room

Solve in the same way as before

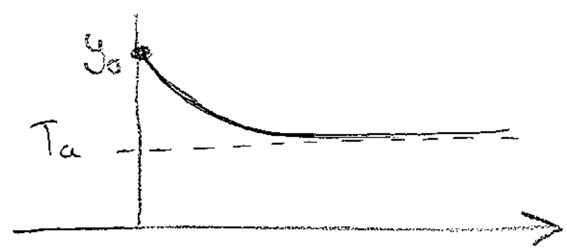
$$\int \frac{y'(t)}{y(t)-T_a}\,dt = \int k\,dt = kt + C$$

$$\ln(y(t)-T_a) \quad \Longrightarrow \quad y(t) = Ae^{kt} + T_a$$

Looking at $t=0$ gives $y(0) = A + T_a$, so $A = y_0 - T_a$ where y_0 is the initial temp of the coffee. Thus

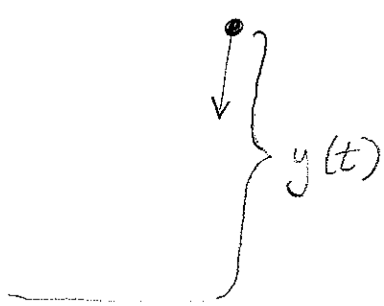
$$y(t) = (y_0 - T_a)e^{kt} + T_a$$

What happens as $t \to \infty$? $y(t) \longrightarrow T_a$



Further Examples: (see also Math 285)

Higher order:



$$y''(t) = -g \quad \Longrightarrow \quad y(t) = y_0 + v_0 t - g t^2$$

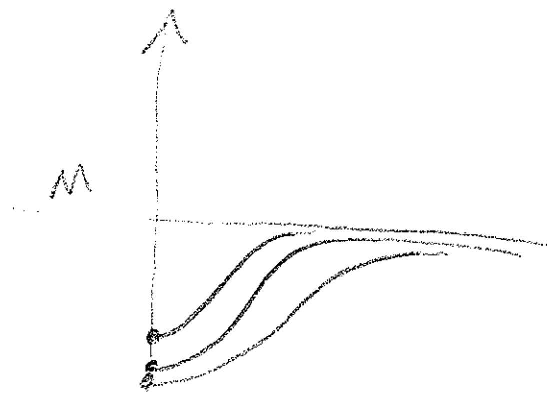
Population growth with limits

carrying capacity

$$p'(t) = k\,p(t)\,(M - p(t))$$

When $p(t)$ is small, looks like $p'(t) = k\,p(t)$, but as $p(t)$ approaches M, $p'(t)$ goes to 0. Leads to

$$p(t) = \frac{AMe^{kMt}}{1 + Ae^{kMt}}$$



As $t \to \infty$, $p(t) \to M$

[Called the logistics equation, used to model world pop.]

If time permits: 1) discrete analogs

2) Partial Differential Equations

3) Sleipner A (80 m / \$700 million / 1991 / 24-cells)