Lecture 10: Review

Extra Office Hours: Wed 2:30-4:30
Thur 2:30 – 5:30

Reminder:
Bring 1 sheet of notes to the exam (handwritten)

Review Exercises: 1–50, 61–68, 70, 71

Partial Fractions: $\displaystyle\int \frac{x^2-x+1}{x^3+2x^2+x}\,dx$

Comment: When deg top $\geq$ deg bottom have to divide first. Not on exam.

Factor Denominator: $x(x^2+2x+1)$

$= x(x+1)^2$

So:

$$\frac{x^2-x+1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$= \frac{A(x+1)^2 + Bx(x+1) + Cx}{x(x+1)^2}$$

To find A, B, C where

$$x^2-x+1 = A(x+1)^2 + Bx(x+1) + Cx = A(x^2+2x+1) + B(x^2+x) + Cx$$

$$= (A+B)x^2 + (2A+B+C)x + A$$

Plug in roots: $x=0$: $1 = A$

$x=-1$: $3 = -C$

Then $x^2-x+1 = (1+B)x^2 + (B-1)x + 1 \Rightarrow B=0$

$$\int \frac{x^2-x+1}{x^3+2x^2+x}\,dx = \int \frac{1}{x} - \frac{3}{(x+1)^2}\,dx$$

$$= \ln x + \frac{3}{x+1} + C.$$

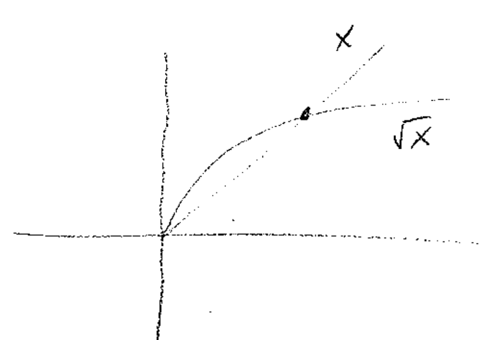
Does $\displaystyle\int_1^\infty \frac{2+\ln x+\cos x}{x+\sqrt{x}+1}\,dx$ converge or diverge?

Hmm, integrand looks like $\frac{\ln x}{x}$ which is $\geq \frac{1}{x}$ for $x \geq e$.
As $\int_1^\infty \frac{1}{x}\,dx$ diverges, we'll guess our integral does too.

Now $2+\ln x+\cos x \geq 1$ as $\ln x \geq 0$ for $x \geq 1$
$\cos x \geq -1$ for any x.

and

$$x+\sqrt{x}+1 \leq 2x+1 \quad \text{as } \sqrt{x} \leq x \text{ for } x \geq \underline{1}.$$



So

$$\frac{1}{2x+1} \leq \frac{2+\ln x+\cos x}{x+\sqrt{x}+1}$$

Now $\displaystyle\int_1^\infty \frac{1}{2x+1}\,dx = \lim_{R\to\infty}\int_1^R \frac{1}{2x+1}\,dx = \lim_{R\to\infty} \frac{1}{2}\ln|2x+1|\Big|_1^R$

$$\lim_{R\to\infty} \frac{1}{2}\left(\ln(2R+1) - \ln 3\right) = +\infty,$$ so integral diverges.

By comparison test, $\displaystyle\int_1^\infty \frac{2+\ln x+\cos x}{x+\sqrt{x}+1}\,dx$ diverges as well.

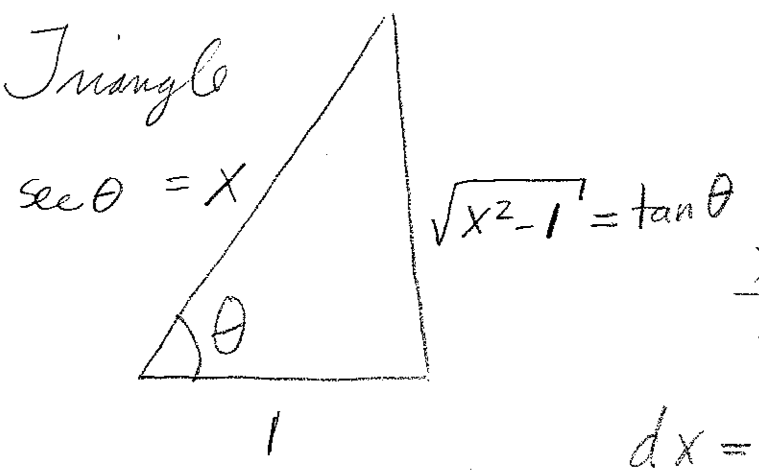
Trig Substitution: $\sqrt{x^2-a^2}$

$\displaystyle\int x^3\sqrt{x^2-1}\,dx$ integrand defined for x in $(-\infty,-1]$ and $[1,\infty)$

Need to pick which interval ↗, say $x \geq 1$.

(Compare $\displaystyle\int \frac{1}{x\sqrt{x^2-1}} = \begin{cases} \sec^{-1}x & \text{if } x\geq 1 \\ -\sec^{-1}x & \text{if } x\leq -1 \end{cases}$ as $\dfrac{d}{dx}\sec^{-1} = \dfrac{1}{|x|\sqrt{x^2-1}}$)

Triangle



Here θ is in $[0, \pi/2)$

$$\frac{x}{1} = \frac{hyp}{adj} = \sec\theta$$

$$dx = \sec\theta\tan\theta\,d\theta$$

do

$$\int x^3\sqrt{x^2-1}\,dx = \int (\sec^3\theta\tan\theta)\sec\theta\tan\theta\,d\theta$$

$$= \int \sec^4\theta\tan^2\theta\,d\theta$$

Two approaches:

$u = \sec\theta$
$du = \sec\theta\tan\theta\,d\theta$

or

$u = \tan\theta$
$du = \sec^2\theta\,d\theta$

$$= \int \underbrace{\sec^2\theta}_{1+\tan^2\theta}\tan^2\theta\,\underbrace{\sec^2\theta\,d\theta}_{du}$$

$$= \int (1+u^2)u^2\,du = \frac{1}{3}u^3 + \frac{1}{5}u^5 + C \qquad u = \tan\theta = \sqrt{x^2-1}$$

$$= \frac{1}{3}(x^2-1)^{3/2} + \frac{1}{5}(x^2-1)^{5/2} + C.$$

Others (if time remains):

$$\int \tan^3 x\,dx = \int (\sec^2 x - 1)\tan x\,dx$$

$$\int e^{6x}\sin e^{2x}\,dx$$

$$\int \frac{1}{x^4-x}\,dx \qquad u = x^3+1 \qquad u = 1/x$$