Lecture 8: Partial Fractions (§6.4)

HW #4: (Sept 17) §6.6 41, 45, 46
§6.4 5, 6

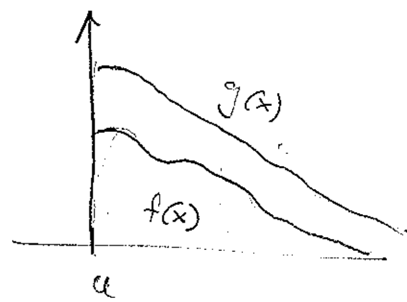
Last time:

Comparison Test: f, g continuous on $[a, \infty)$, and

$$0 \leq f(x) \leq g(x) \quad \text{for all } x \text{ in } [a, \infty)$$

a) If $\int_a^\infty g(x)\,dx$ converges, so does $\int_a^\infty f(x)\,dx$.

b) If $\int_a^\infty f(x)\,dx$ diverges, so does $\int_a^\infty g(x)\,dx$

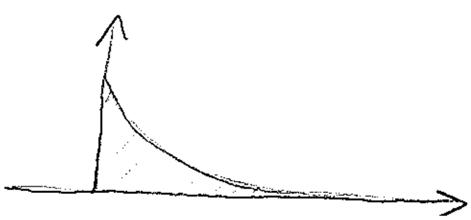


[Also works for $(-\infty, a]$ or $[a, b]$.]

Ex: $\displaystyle\int_0^\infty \frac{1}{x^2+x^5+e^x}\,dx$

Focusing on the bottom

$$x^2+x^5+e^x$$



the most important term is e^x

Suggests comparison with $1/e^x$

Indeed, $x^2+x^5+e^x \geq e^x$ and so

$$\frac{1}{e^x} \geq \frac{1}{x^2+x^5+e^x} \geq 0$$

As $\int_0^\infty 1/e^x\,dx$ converges, so does $\displaystyle\int_0^\infty \frac{1}{x^2+x^5+e^x}\,dx$.

Partial Fractions: (§6.4)

For things like $\int \frac{2x+3}{x^2+3x+2}\,dx$, partial fractions is a way to algebraically manipulate the integrand, into pieces such as $\int \frac{1}{x+1}\,dx = \ln|x+1| + C$.

Point: $$\frac{1}{x+1} + \frac{1}{x+2} = \frac{x+2}{(x+1)(x+2)} + \frac{x+1}{(x+1)(x+2)}$$
$$= \frac{2x+3}{x^2+3x+2}$$

Hence $$\int \frac{2x+3}{x^2+3x+2}\,dx = \int \underbrace{\frac{1}{x+1} + \frac{1}{x+2}}\,dx = \ln|x+1| + \ln|x+2| + C.$$

The partial fraction decomposition is a way to find from .

Ex: $\int \frac{1}{x^2+x-2}\,dx$ First notice that the denominator factors: $x^2+x-2 = (x-1)(x+2)$

Now $$\frac{1}{x-1} + \frac{1}{x+2} = \frac{(x+2)+(x-1)}{(x-1)(x-2)} = \frac{2x+1}{x^2+x-2}$$

Hmm, that wasn't so helpful. However, let's try

$$\frac{A}{x-1} + \frac{B}{x+2} = \frac{A(x+2) + B(x-1)}{(x-1)(x-2)} = \frac{(A+B)x + (2A-B)}{(x-1)(x-2)}$$

for some constants A & B. For this to be $\frac{1}{(x-1)(x-2)}$

we must have $\left.\begin{aligned} A+B &= 0 \\ 2A-B &= 1 \end{aligned}\right\} \rightarrow A = -B$ and $3A = 1$,

i.e. $A = 1/3$, $B = -1/3$.

__Check__: $\frac{1}{3}\frac{1}{(x-1)} - \frac{1}{3}\frac{1}{x+2} = \frac{1}{(x-1)(x-2)}$ ✓

__So__: $\int \frac{1}{x^2+x-2}\,dx = \int \frac{1}{3}\left(\frac{1}{x-1} - \frac{1}{x+2}\right)dx = \frac{1}{3}\Big(\ln|x-1| - \ln|x+2|\Big) + C$

__General Linear Factors:__

degree $< n$

$$\frac{P(x)}{\underbrace{(a_1x+b_1)(a_2x+b_2)\cdots(a_nx+b_n)}_{\text{distinct}}} = \frac{c_1}{a_1x+b_1} + \cdots + \frac{c_n}{a_nx_n+b_n}$$

for some $c_1, \ldots, c_n$

Ex: $\dfrac{3x^2-7x-2}{x^3-x}$ Notice $x^3-x = x(x-1)(x+1)$ and then set

$$\frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} = \frac{A\overset{x^2-1}{(x-1)(x+1)} + B\overset{x^2+x}{x(x+1)} + C\overset{x^2-x}{x(x-1)}}{x(x-1)(x+1)}$$

$$= \frac{(A+B+C)x^2 + (B-C)x + (-A+B+C)}{x^3-x}$$

That is
$$A+B+C=3$$
$$B-C=-7$$
$$-A+B+C=-2$$

Could just solve, but here's a quicker way:

$$\longrightarrow \quad 3x^2-7x-2 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$$

Let $x=0$, find $-2=-A \implies A=2$

$x=1$, find $-6=2B \implies B=-3$

$x=-1$, find $8=2C \implies C=4$

and so

$$\frac{3x^2-7x-2}{x^3-x} = \frac{2}{x} + \frac{-3}{x-1} + \frac{4}{x+1}$$