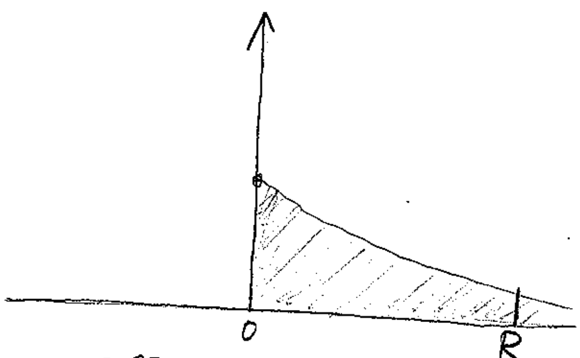
Lecture 7: Improper integrals, II (§6.6)

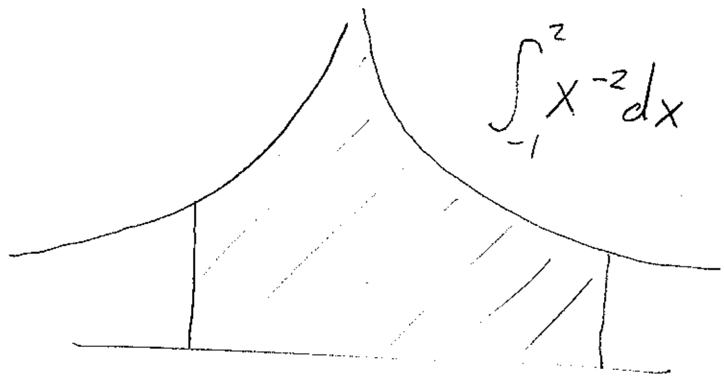
HW #4: Due Sept 17 §6.6 11, 18, 25, 35, 39, 40, 54–56

Math Dept Tutoring: MTuWTh from 7–9:30 in 2 Illini Hall

Revised Honors Problem set:

Improper Integrals: Those where the Fund Thm of Calc does not directly apply.





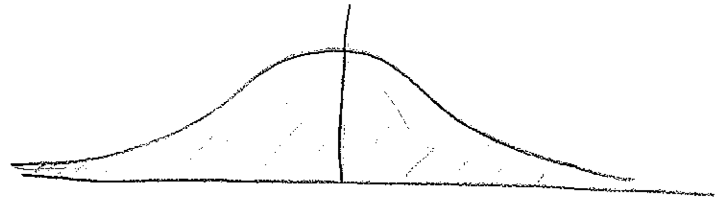
$$\int_{-1}^{2} x^{-2}\,dx$$

$$\int_0^\infty e^{-x}\,dx = \lim_{R\to\infty} \int_0^R e^{-x}\,dx = \lim_{R\to\infty} -e^{-x}\Big|_0^R$$

$$= \lim_{R\to\infty} -e^{-R} + e^0 = 1$$

Can also look at integrals over the whole x-axis

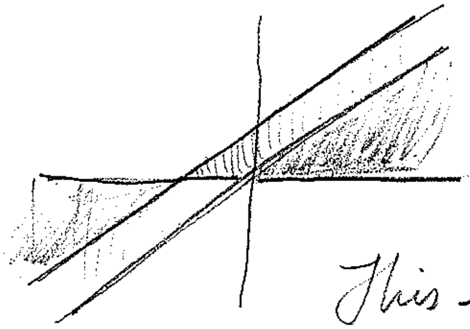
$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$



Q: How should we define $\int_{-\infty}^{\infty} f(x)\,dx$?

Try 1: $\lim\limits_{R\to\infty} \int_{-R}^{R} f(x)\,dx$

Not quite right: $\int_{-\infty}^{\infty} x\,dx \overset{\text{not really}}{=} \lim\limits_{R\to\infty} \int_{-R}^{R} x\,dx$



$$= \lim_{R\to\infty} \left.\frac{x^2}{2}\right|_{-R}^{R} = \lim_{R\to\infty} 0 = 0.$$

This isn't unreasonable, but what about $\underbrace{\int_{-\infty}^{\infty} x+1}\,dx = \lim\limits_{R\to\infty} \int_{-R}^{R} x+1\,dx$

same graph as x, but shifted over to the left, so answer should be the same, i.e. 0.

$$= \lim_{R\to\infty} \left.\frac{x^2}{2}+x\right|_{-R}^{R} = \lim_{R\to\infty} 2R$$

which does not exist.

Correct: If f is cont on $(-\infty,\infty)$, we write

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx \quad \text{for any constant } a,$$

and say converges if and only if both of the righthand integrals converge.

Ex: $\int_{-\infty}^{\infty} x\,dx = \int_{-\infty}^{0} x\,dx + \int_{0}^{\infty} x\,dx$

diverges as $\int_{-\infty}^{0} x\,dx$ does.

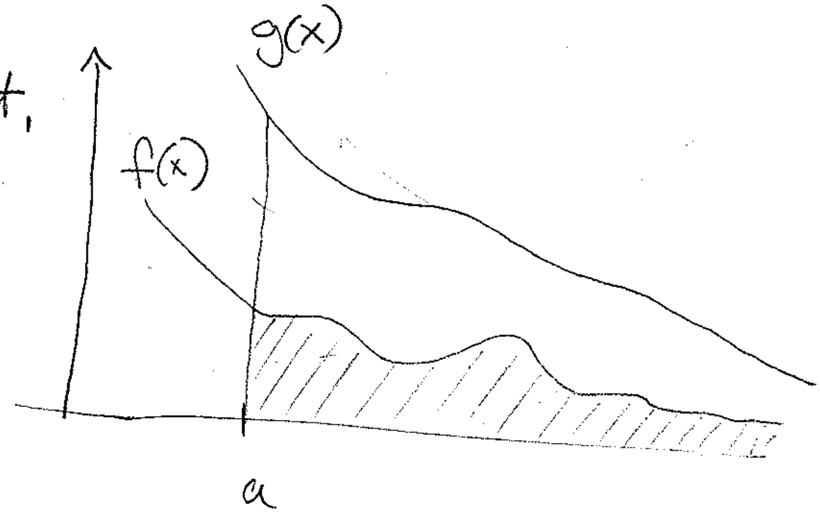
Comparison Test: Deciding when $\int_0^\infty f(x)\,dx$ converges, even when we can't explicitly compute it.

Ex: $\int_2^\infty e^{-x^2}dx$, where we don't have a closed form for $\int e^{-x^2}dx$

General method: f, g cont,

$0 \le f(x) \le g(x)$

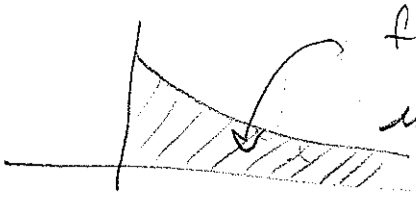
for x in $[a, \infty)$



Then if

$\int_a^\infty g(x)\,dx$ converges, so does $\int_0^\infty f(x)\,dx$

Idea: For a function $h(x) \ge 0$, then



finite area $\Longleftrightarrow$ $\int_a^\infty h\,dx$ converges

infinite area $\Longleftrightarrow$ $\int_a^\infty h\,dx$ diverges

means "if and only if"

As the graph of f lies below that of g, clearly the area under f is $\leq$ area under g.

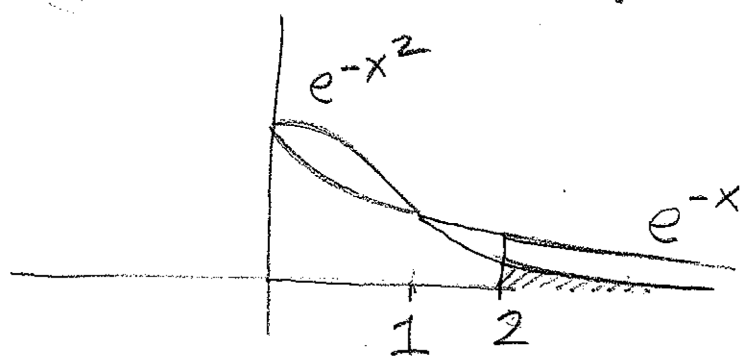
So if the area under g is finite, so is the area under f.

Similarly: if $\int_a^\infty f(x)\,dx$ diverges, so does $\int_a^\infty g(x)\,dx$.

Ex: $\int_2^\infty e^{-x^2}\,dx$ On $[2,\infty)$ we have $x < x^2$,
so $e^x < e^{x^2}$ so $e^{-x^2} < e^{-x}$.

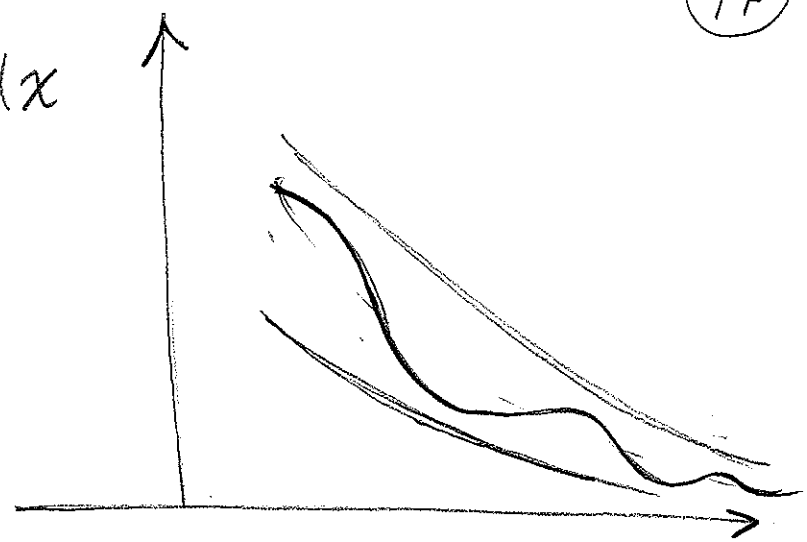
As $\int_1^\infty e^{-x}\,dx$ converges,

so does $\int_1^\infty e^{-x^2}\,dx$.



[Can also apply to $\int_0^\infty e^{-x^2}\,dx$ by noting that this is equal to $\int_0^2 e^{-x^2}\,dx + \int_2^\infty e^{-x^2}\,dx$.]

Ex: $\displaystyle\int_1^\infty \frac{3+\cos x}{x}\,dx$



Now $-1 \le \cos x \le 1$,

so

$$\frac{2}{x} \le \frac{3+\cos x}{x} \le \frac{4}{x}$$

As $\displaystyle\int_1^\infty \frac{2}{x}\,dx$ diverges, so does $\displaystyle\int_1^\infty \frac{3+\cos x}{x}\,dx$