Lecture 18: Alternating Series (§8.4) / General Series (§8.5)

HW #7: Oct 15: §8.4 #: 5, 6, 19, 24, 29, 37

Next time: More on §8.5.

Last time: Alternating Series: $\sum_{k=0}^{\infty}(-1)^k a_k = a_0 - a_1 + a_2 - a_3 + \cdots$ where $a_k \geq 0$.

Ex: $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots = \ln 2$

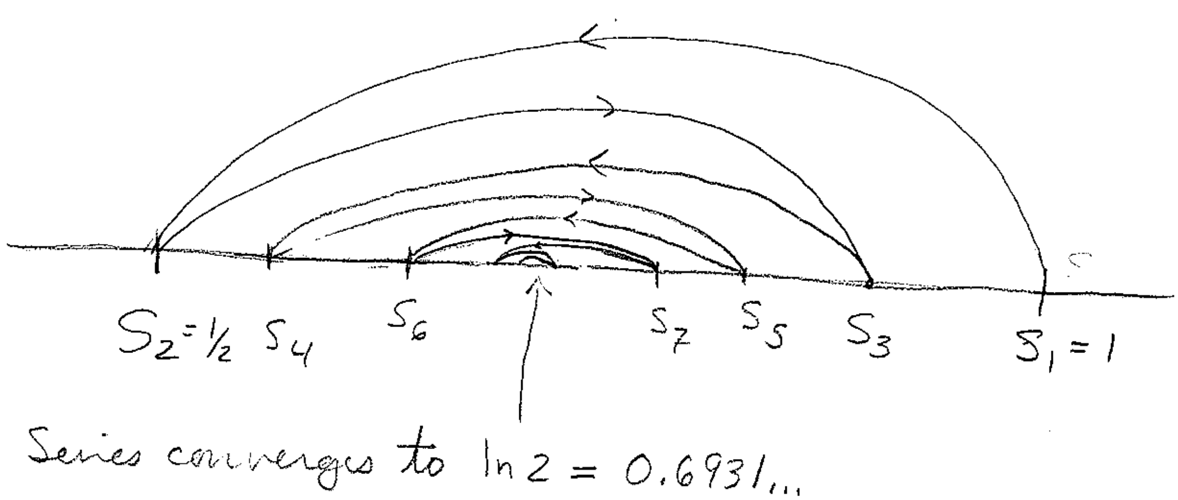
Partial Sums: $\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k}$

$S_1 = 1 \qquad S_2 = S_1 - \frac{1}{2} = 1/2 = 0.5$

$S_3 = S_2 + \frac{1}{3} = 5/6 = .833\ldots \qquad S_4 = S_3 - \frac{1}{4} = 7/12 = 0.5833\ldots$

$S_5 = S_4 + 1/5 = \frac{47}{60} = .7833\ldots \qquad S_6 = S_5 - \frac{1}{6} = 37/60 = 0.6166\ldots$

[Start here:]

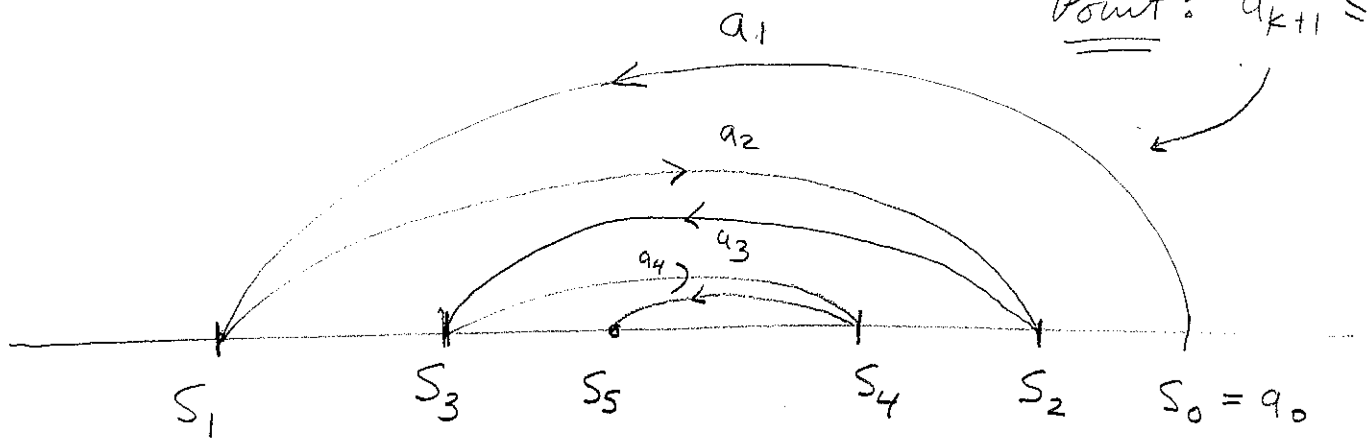


Alternating Series Test: Suppose $0 < a_{k+1} \leq a_k$ for all $k \geq 0$, and $\lim_{k\to\infty} a_k = 0$. Then $\sum_{k=0}^{\infty}(-1)^k a_k$ converges.

Idea: $\sum_{k=0}^{\infty} (-1)^k a_k = a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + \cdots$

$$S_n = \sum_{k=0}^{n} (-1)^k a_k$$

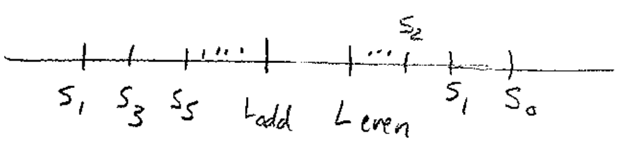
Point: $a_{k+1} \le a_k$



Notice: $S_1 \le S_3 \le S_5 \le S_7 \le \cdots$ $\quad \{S_{2j+1}\}_{j=0}^{\infty}$ – increasing

$S_0 \ge S_2 \ge S_4 \ge S_6 \ge \cdots$ $\quad \{S_{2j}\}_{j=0}^{\infty}$ – decreasing

Also $S_1 \le S_n \le S_0$ for all n. As bounded monotone sequences converge, $\lim_{j\to\infty} S_{2j+1} = L_{odd}$



$\lim_{j\to\infty} S_{2j} = L_{even}$

Now

$$0 = \lim_{j\to\infty} a_{2j+1} = \lim_{j\to\infty} S_{2j+1} - S_{2j} = \lim_{j\to\infty} S_{2j+1} - \lim S_{2j}$$

$$= L_{odd} - L_{even}.$$

Thus $L_{odd} = L_{even}$ and $\lim_{n\to\infty} S_n$ exists. So $\sum_{k=1}^{\infty} a_k$ converges.

<u>Error bounds:</u> Suppose the Alt. Series Test says that

$\sum_{k=0}^{\infty} (-1)^k a_k$ converges. Then $\underbrace{\left| \sum_{k=0}^{\infty} (-1)^k a_k - \sum_{k=1}^{n} (-1)^k a_k \right|}_{\text{remainder } R_n} \leq a_{n+1}$

<u>Ex:</u> $\sum_{k=1}^{10{,}000} \frac{(-1)^k}{k} = 0.6931971\ldots$

is within $a_{10{,}001} = \frac{1}{10{,}001}$ of $\sum_{k=1}^{\infty} \frac{(-1)^k}{k} = 0.6931471\ldots$

<u>General series:</u> (§8.5) [Recap story so far.]

$$\sum_{k=1}^{\infty} \frac{\cos k}{k^2} = \underbrace{\cos 1}_{>0} + \underbrace{\tfrac{1}{4}\cos 2 + \tfrac{1}{9}\cos 3 + \tfrac{1}{16}\cos 4}_{<0} + \underbrace{\tfrac{1}{25}\cos 5 +}_{>0}$$

<u>Absolute Convergence:</u>

$\sum_{k=1}^{\infty} a_k$ converges absolutely if $\sum_{k=1}^{\infty} |a_k|$ converges.

<u>Point:</u> If $\sum_{k=1}^{\infty} a_k$ is absolutely convergent, then it converges.

[Give rough idea of why, careful explanation for next time. Useful as we now have many tools for dealing with pos. series.]

Ex: $\sum_{k=1}^{\infty} \frac{\cos k}{k^2}$ Consider $\sum_{k=1}^{\infty} \left| \frac{\cos k}{k^2} \right|$.

Now this converges, since $\left| \frac{\cos k}{k^2} \right| \le \frac{1}{k^2}$ and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges.

So $\uparrow$ is absolutely convergent, hence converges.

$$\sum_{k=1}^{\infty} \frac{\cos k}{k^2} = 0.324\ldots \qquad \text{vs.} \qquad \sum_{k=1}^{\infty} \left| \frac{\cos k}{k^2} \right| = 0.927\ldots$$

Ex: $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ converges, but is not absolutely convergent since $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges

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Why absolute convergence implies convergence:

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} a_k + |a_k| - |a_k|$$

$$= \underbrace{\sum_{k=1}^{\infty} \underbrace{a_k + |a_k|}_{\geq 0}}_{\text{converges by comparison with}} - \sum_{k=1}^{\infty} |a_k| \Bigg\} \text{ converges by assumption}$$

$\Longrightarrow$ So $\sum_{k=1}^{\infty} a_k$ converges

$\sum_{k=1}^{\infty} 2|a_k|$ which converges as $\sum_{k=1}^{\infty} |a_k|$ does