Lecture 17: More convergence tests. (§8.3)

Alternating series (§8.4)

HW #6 Due Oct 8: §8.3: # 21, 25, 26, 31, 34

Note: This weeks Tuesday office hours moved to 4–5.

Next time: Rest of §8.4, start §8.5.

Last time: $\sum_{k=1}^{\infty} a_k$ a series with $a_k \geq 0$. [Start w/ quick fact on next page.]

Integral Test: If $a_k = f(k)$ for a continuous decreasing function f with $f(x) \geq 0$. Then $\sum_{k=1}^{\infty} a_k$ and $\int_1^{\infty} f(x)\,dx$ both converge or both diverge.

Comparison test: $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ where $0 \leq a_k \leq b_k$.

If $\sum_{k=1}^{\infty} b_k$ converges, so does $\sum_{k=1}^{\infty} a_k$.

If $\sum_{k=1}^{\infty} a_k$ diverges, so does $\sum_{k=1}^{\infty} b_k$.

Idea: $\sum_{k=1}^{\infty} a_k \leq \sum_{k=1}^{\infty} b_k$ since $\underbrace{\sum_{k=1}^{n} a_k}_{a_1+a_2+\cdots+a_n} \leq \underbrace{\sum_{k=1}^{n} b_k}_{b_1+b_2+\cdots+b_n}$.

Ex: $\sum_{k=1}^{\infty} \frac{1}{k^2+5}$ converges since $\frac{1}{k^2+5} \leq \frac{1}{k^2}$ and $\sum_{k=1}^{\infty} 1/k^2$ converges.

Non-Ex: $\sum_{k=3}^{\infty} \frac{1}{k^2-5}$ Should converge as $\frac{1}{k^2-5}$ is "basically" $\frac{1}{k^2}$

But $\frac{1}{k^2-5} \geq \frac{1}{k^2}$ so can't directly apply the Comp. Test.

Limit Comparison Test: $a_k, b_k > 0$. Suppose $\lim_{k\to\infty} \frac{a_k}{b_k} = L$ where $L \neq 0$ or ∞. Then $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ both converge or both diverge.

Ex: $\sum_{k=3}^{\infty} \frac{1}{k^2-5}$ Now $\lim_{k\to\infty} \frac{\frac{1}{k^2-5}}{1/k^2} = \lim_{k\to\infty} \frac{k^2}{k^2-5} = 1$

So since $\sum_{k=1}^{\infty} 1/k^2$ converges, so does $\sum_{k=1}^{\infty} \frac{1}{k^2-5}$

Idea behind LCT: For some N, we have $\left|\frac{a_k}{b_k} - L\right| < \frac{L}{2}$.
That is, $\frac{1}{2}L < \frac{a_k}{b_k} < \frac{3}{2}L$ and so $\left(\frac{1}{2}L\right) b_k \leq a_k \leq \left(\frac{3}{2}L\right) b_k$
Now $\sum_{k=1}^{\infty} (1/2\, L) b_k$ and $\sum_{k=1}^{\infty} \left(\frac{3}{2}L\right) b_k$ converge if and only if $\sum_{k=1}^{\infty} b_k$. Now apply the Comparison Test.

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<u>Quick Fact:</u> $\sum_{k=1}^{\infty} a_k$, $\sum_{k=1}^{\infty} b_k$ where a_k, b_k can be < 0.

If both converge, $\sum_{k=1}^{\infty}(a_k + b_k)$ converges to $\sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k$

If one converges and the other diverges, $\sum_{k=1}^{\infty}(a_k + b_k)$ diverges

[If they both diverge, then it depends; see honors HW.]

<u>Alternative Series:</u> [Motivate.]

$$\sum_{k=0}^{\infty} (-1)^k \frac{1}{2k+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots = \frac{\pi}{4}$$

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots$$

$$= \ln 2 = 0.693147\ldots$$

Notice that converges, even though $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

<u>General Form:</u>

$$\sum_{k=0}^{\infty} (-1)^k a_k = a_0 - a_1 + a_2 - a_3 + \cdots$$

or

$$\sum_{k=1}^{\infty} (-1)^{k+1} a_k = a_1 - a_2 + a_3 - a_4$$

Alternating Series Test:

Suppose $0 < a_{k+1} \leq a_k$ for all $k \geq 0$ and $\lim_{k \to \infty} a_k = 0$. Then $\sum_{k=1}^{\infty} (-1)^k a_k$ converges.

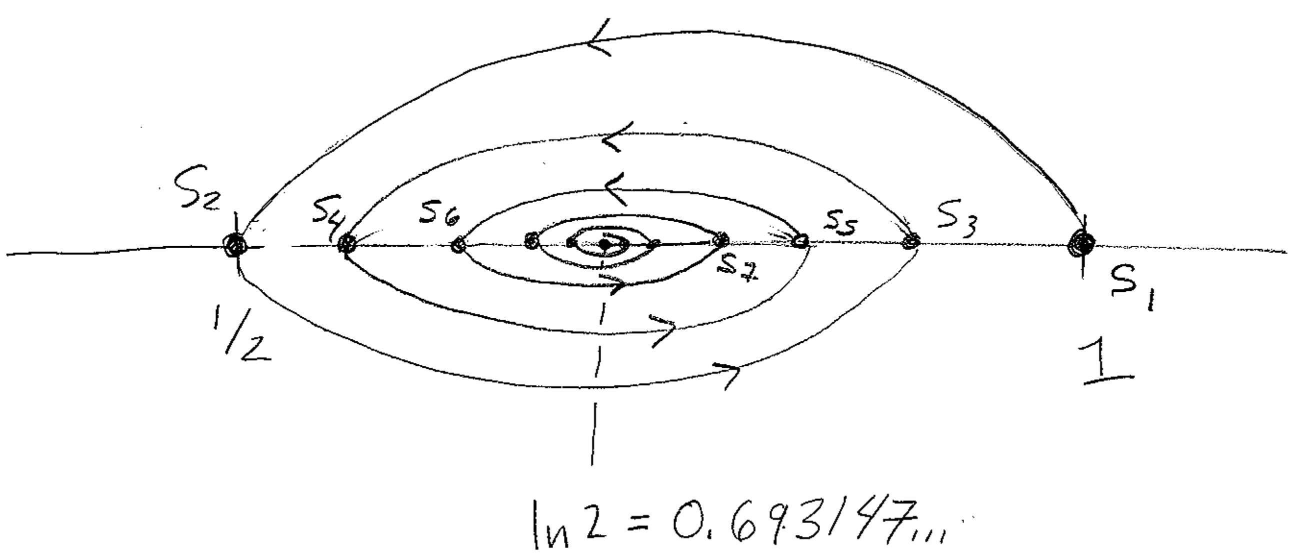
Ex: Thus the two initial examples converge, since $\frac{1}{k}, \frac{1}{2k+1}$ are decreasing, $\to 0$ as $k \to \infty$

Look at: $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$ $\qquad S_n = \sum_{k=1}^{n} (-1)^{k+1} \frac{1}{k}$

$S_1 = 1$ $\qquad S_2 = 1 - 1/2 = 1/2$

$S_3 = S_2 + 1/3 = 5/6 = .833\ldots$ $\qquad S_4 = S_3 - 1/4 = \frac{7}{12} = 0.5833\ldots$

$S_5 = S_4 + 1/5 = \frac{47}{60} = .7833$ $\qquad S_6 = S_5 - 1/6 = \frac{37}{60} = 0.6166\ldots$



$\ln 2 = 0.69314 7\ldots$