Lecture 26: Applications of Taylor Series / Fourier Series

HW (Nov 5): §8.7 : # 28, 47
§8.8 : # 1, 4, 25, 26, 27.

Next time: Fourier Series

Estimating Error: Two ways: [Taylor's Theorem / Alt. series test.]

★ Estimate: $\ln(1.1)$

From HW, you know $\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k$

$= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 \ldots$ for $|x| < 1$.

So

$$\ln(1+0.1) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left(\frac{1}{10}\right)^k$$

$$= \underbrace{\frac{1}{10} - \frac{1}{200} + \frac{1}{3000} - \frac{1}{40000}} + \frac{1}{500{,}000} - \ldots$$

By A.S.T, $0.095308333\ldots$ is within $\frac{1}{500{,}000} = 2\times 10^{-6}$

of $\ln(1.1) = 0.095310798\ldots$

★ Estimate: $\ln(0.9)$. Well,

$$\ln(1 - \tfrac{1}{10}) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\left(-\frac{1}{10}\right)^k = \sum_{k=1}^{\infty} -\frac{1}{k}\left(\frac{1}{10}\right)^k$$

$$= \underbrace{-\frac{1}{10} - \frac{1}{200} - \frac{1}{3000} - \frac{1}{40000}}_{-0.105358333\ldots} - \frac{1}{500{,}000}$$

Here, all terms have the same sign, so can't use the A.S.T. But have Taylor's Theorem:

$$\boxed{f(x) - P_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}}$$ for some z between c and x.

In our case, the error term is $\frac{f^{(5)}(z)}{5!}\left(-\frac{1}{10}\right)^5$

for some z between 0 and $-\frac{1}{10}$. Now $f^{(5)}(z) = 4!\,(z+1)^{-5}$. So

$$|\text{Error}| = \frac{1}{5}\left(\frac{1}{10}\right)^5 \underbrace{\frac{1}{(z+1)^5}}_{\leq \frac{1}{(1-\frac{1}{10})^5} = \left(\frac{10}{9}\right)^5} \leq \frac{1}{5}\left(\frac{1}{10}\right)^5\left(\frac{10}{9}\right)^5 = \frac{1}{5\cdot 9^5} = 3.4\times 10^{-6}$$

since z is in $\left[-\frac{1}{10}, 0\right]$

Note: Where you center the Taylor series matters.

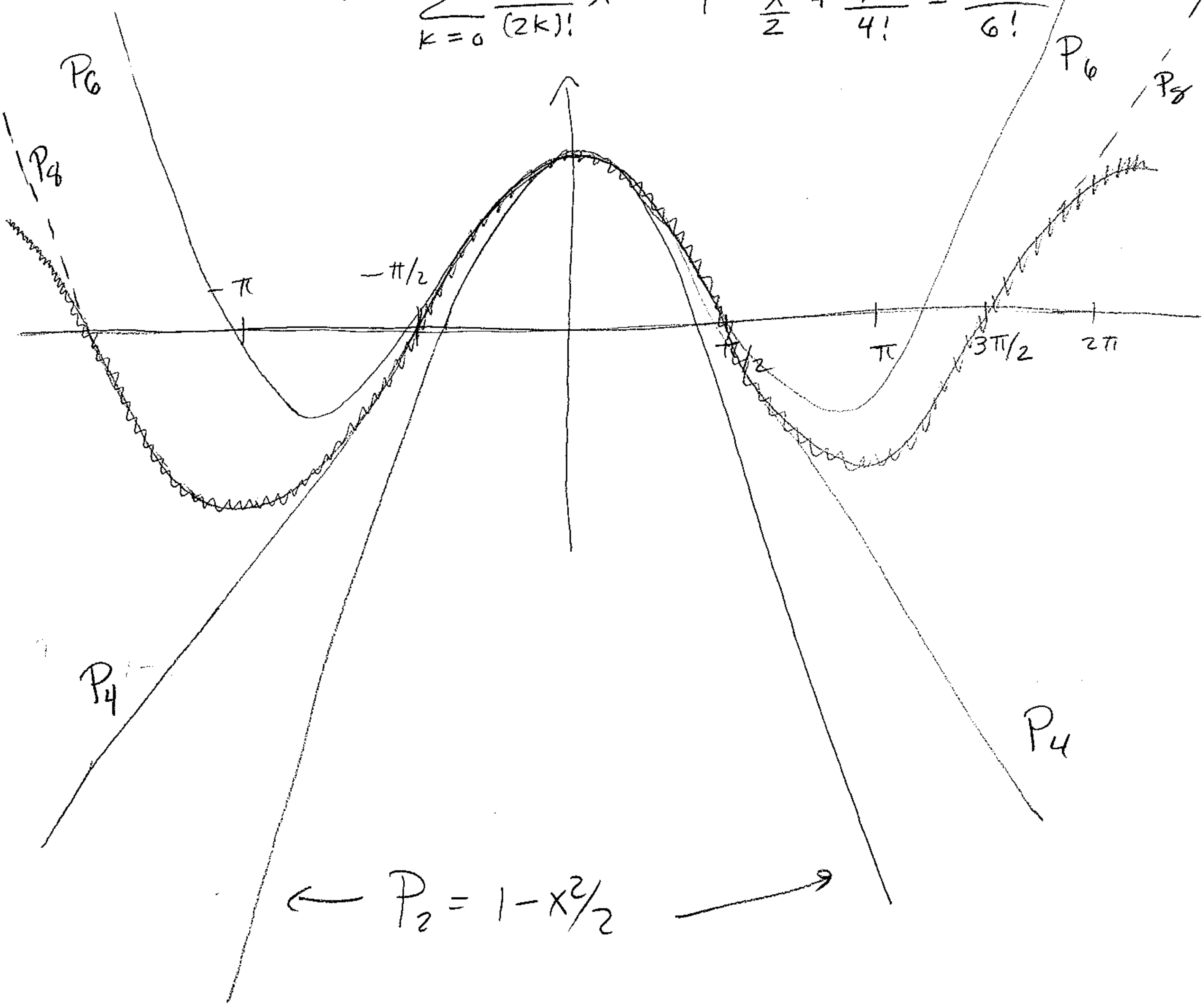
$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}x^{2k+1} = \underbrace{\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\left(x-\frac{\pi}{2}\right)^{2k}}_{\text{see text.}}$$

If we want to compute $\sin(1.23)$ the 2^{nd} series works better: Eg. if desired accuracy is 10^{-11} need 8 terms with 1^{st} series, but only 5 with the second

Point: 1,23 is closer to $\pi/2 = 1.57...$ than 0.

For a repeating thing like sin or cos, the Taylor polynomials are always poor approximates for large x.

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!}$$



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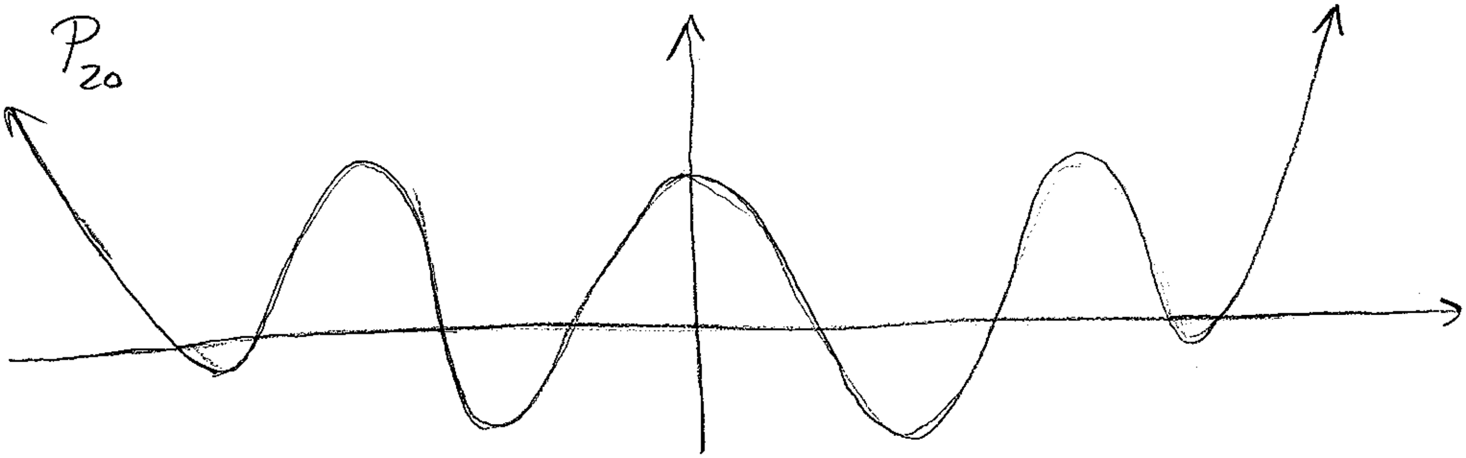
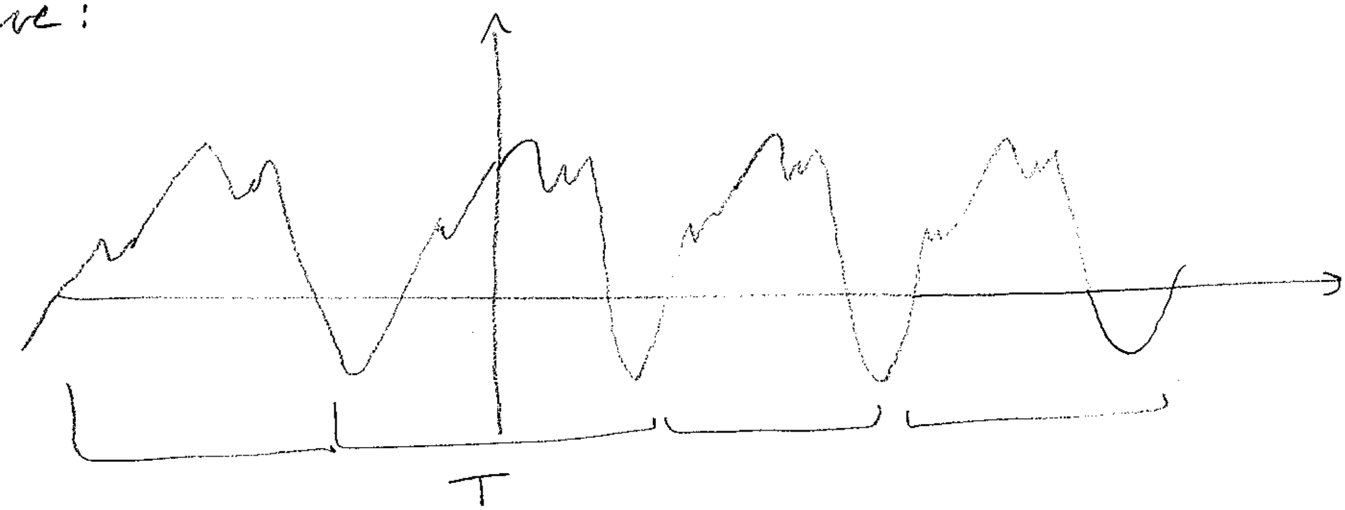


Have the same issue with any periodic wave:



To analyze such things we have another tool: Fourier Series (§8.9)

A function is periodic of period T if $f(x+T) = f(x)$ for all x in the domain of f.

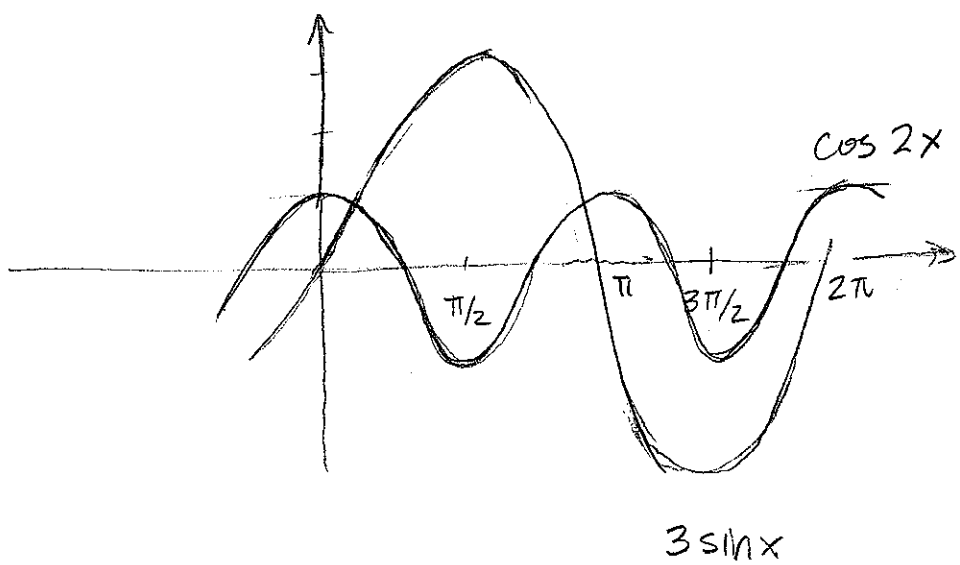
Ex: $\sin x$, $\cos x$ are periodic with period 2π.

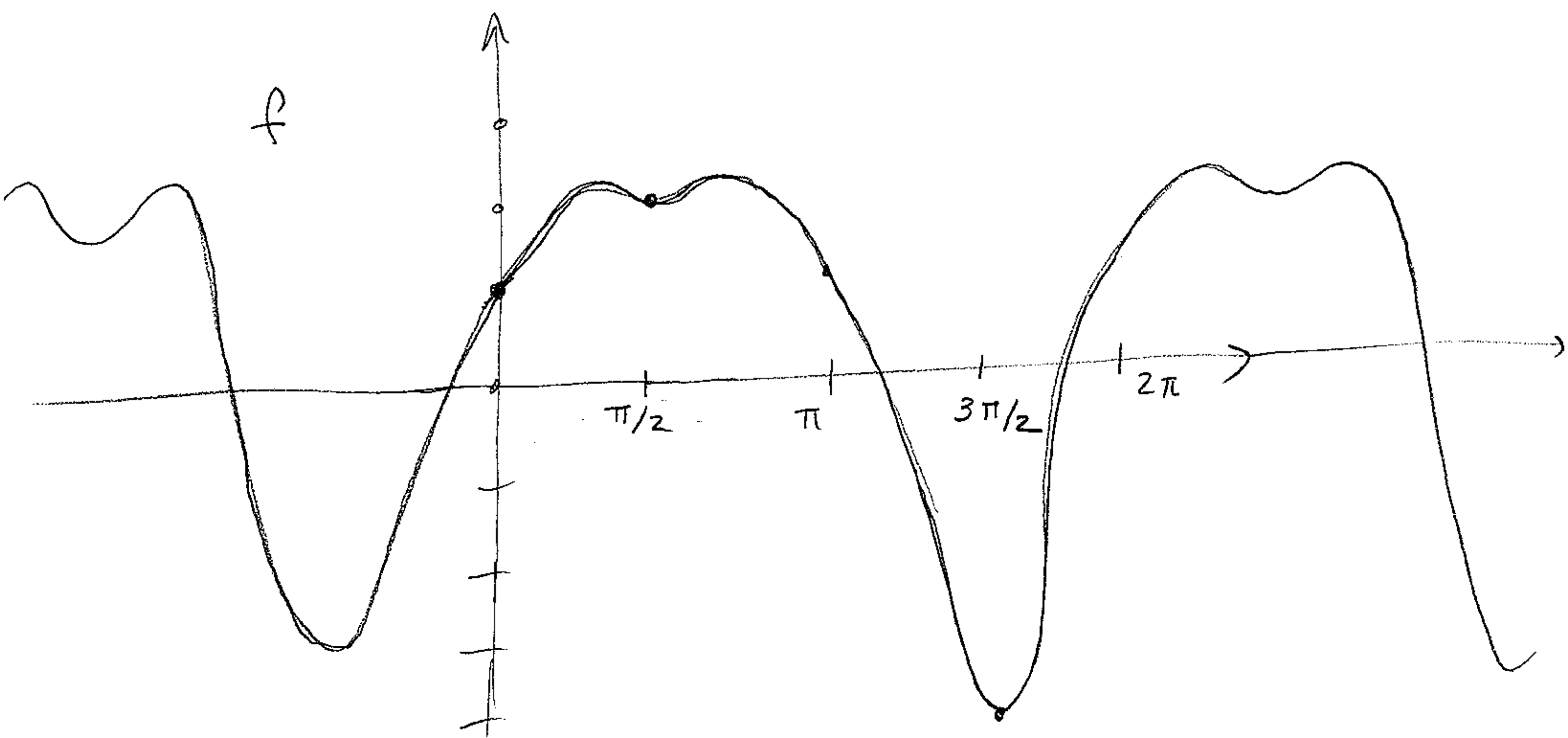
A Fourier series is something of the form

$$\frac{a_0}{2} + \sum_{k=0}^{\infty} \left(a_k \cos(kx) + b_k \sin(kx) \right)$$

for some numbers a_k and b_k.

Ex: $3 \sin x + \cos 2x = f(x)$





Basic questions: ① Given a_k, b_k does this Fourier series make sense?

② Given f which is 2π-periodic, can we express it as a Fourier series.
If so, how do we find the a_k and b_k?

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx$$