Lecture 16: Infinite series with positive terms (§8.3)

HW #6 Due Oct 8: § 8.3 # 7, 8, 9, 15, 43, 49

Next time: Rest of §8.3, §8.4

Last time: $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{k \to \infty} a_k = 0$

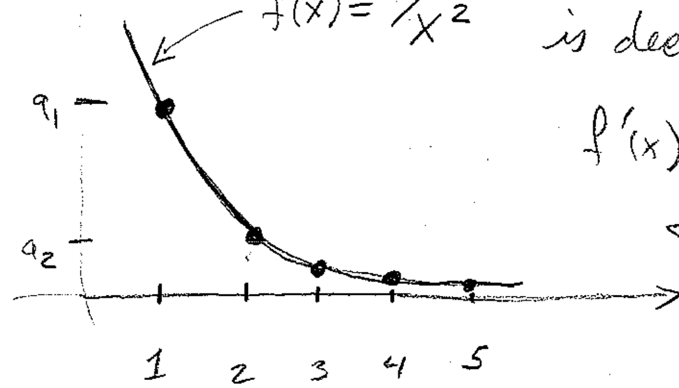
But: $\sum_{k=1}^{\infty} 1/k$ diverges $\rightsquigarrow$ Q: How to decide when a series converges?

For now, let's focus on series $\sum_{k=1}^{\infty} a_k$ when $a_k \geq 0$.

[Explain why easier.]

Integral test: Suppose $a_k = f(k)$ for $k = 1, 2, 3, 4, \ldots$ where f is continuous decreasing function with $f(x) \geq 0$. Then $\sum_{k=1}^{\infty} a_k$ and $\int_1^{\infty} f(x)\,dx$ both converge or both diverge.

Ex: $\sum_{k=1}^{\infty} \frac{1}{k^2}$



$f(x) = 1/x^2$ is decreasing as $f'(x) = -2x^{-3} < 0$ for $x \geq 1$.

$$\int_1^{\infty} 1/x^2\,dx = \lim_{R \to \infty} \int_1^R x^{-2}\,dx$$

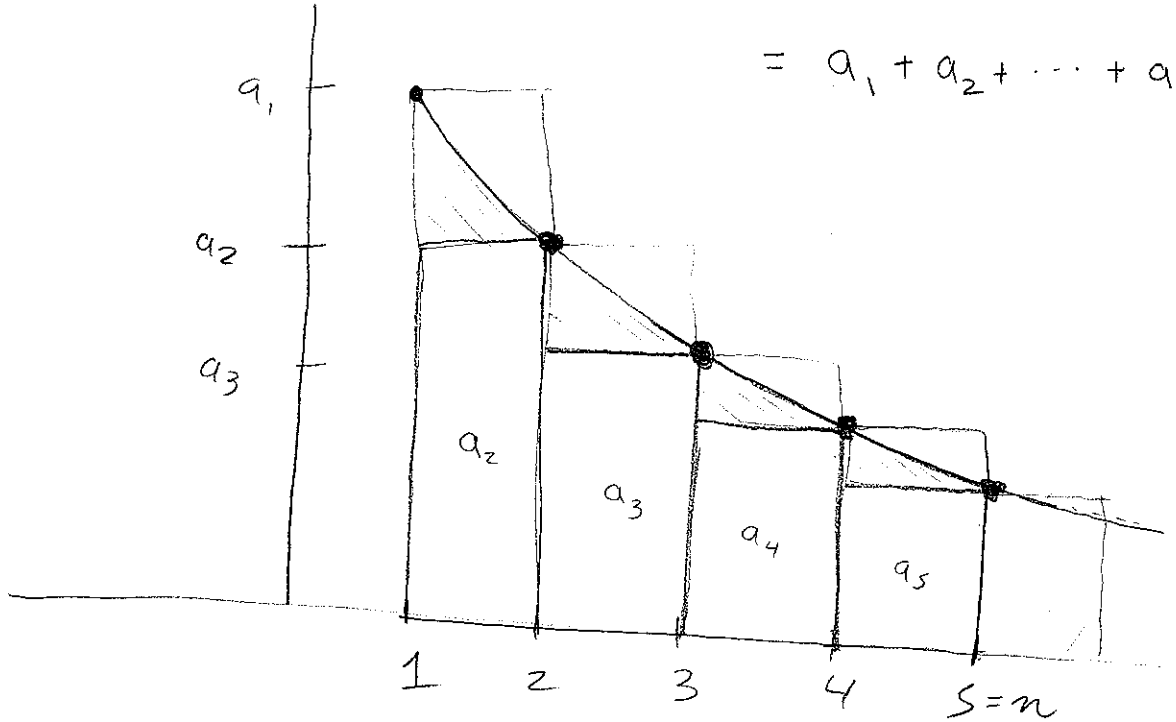
$$\lim_{R \to \infty} -x^{-1}\Big|_1^R = 1.$$

$\to$ So by Integral Test, $\sum_{k=1}^{\infty} 1/k^2$ converges

Idea behind Integral Test:

$$S_n = \sum_{k=1}^{n} a_k$$

$$= a_1 + a_2 + \cdots + a_n$$



Now

$$S_n - a_1 \le \int_1^n f(x)\,dx$$

and

$$\int_1^{n+1} f(x) \le S_n$$

So

$$\int_1^{n+1} f(x)\,dx \le S_n \le a_1 + \int_1^n f(x)\,dx$$

If $\int_1^\infty f(x)\,dx$ converges, then $S_n \le a_1 + \int_1^\infty f(x)\,dx$.

Then, $\{S_n\}$ is a bounded increasing sequence, hence converges. So $\sum_{k=1}^{\infty} a_k$ converges.

Conversely, if $\int_1^{\infty} f(x)\,dx$ diverges, then $S_n \to \infty$ and so $\sum_{k=1}^{\infty} a_k$ diverges.

<u>Note</u>: When things converge, we get

$$\int_1^{\infty} f(x)\,dx \le \sum_{k=1}^{\infty} a_k \le a_1 + \int_1^{\infty} f(x)\,dx$$

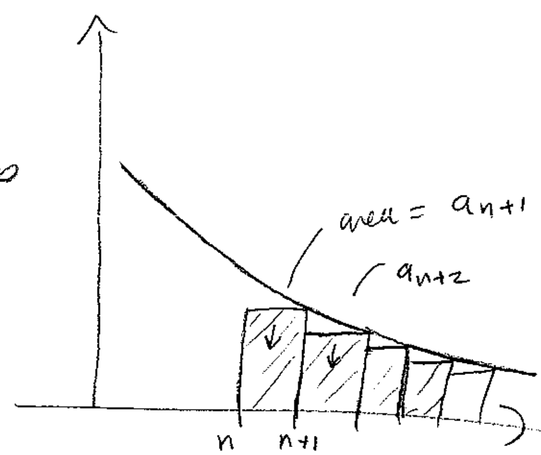
so in our example, $1 \le \sum_{k=1}^{\infty} 1/k^2 \le 2$. Indeed, $\sum_{k=1}^{\infty} 1/k^2 \approx 1.64$

For any series, let $R_n = \sum_{k=n+1}^{\infty} a_k$ so that

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{n} a_k + R_n$$

Now when the integral test applies

$$0 \le R_n = \sum_{k=n+1}^{\infty} a_k \le \int_n^{\infty} f(x)\,dx$$



For $\sum_{k=1}^{\infty} 1/k^2$, we have

$$R_{1000} \le \int_{1000}^{\infty} 1/x^2\,dx = +\frac{1}{1000}$$

Hence

$$\sum_{k=1}^{1000} 1/k^2 = 1.643934566$$

is within $1/1000$ of $\sum_{k=1}^{\infty} 1/k^2 = \pi^2/6 = 1.644934066$

[Skip unless asked: We also have $\int_{n+1}^{\infty} f(x)dx \leq R_n$,

In our current case $\frac{1}{1001} \leq R_n \leq 1/1000$ Thus

$$\underbrace{\sum_{k=1}^{1000} 1/k^2 + \frac{1}{1001}}_{= 1.64493356} \leq \sum_{k=1} 1/k^2 \leq \sum_{k=1}^{1000} 1/k^2 + 1/1000 = 1.64493456$$]

<u>Comparison Test</u>: $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$

where $0 \leq a_k \leq b_k$

If $\sum_{k=1}^{\infty} b_k$ converges, so does $\sum_{k=1}^{\infty} a_k$

If $\sum_{k=1}^{\infty} a_k$ diverges, so does $\sum_{k=1}^{\infty} b_k$

<u>Idea</u>: $\sum_{k=1}^{\infty} a_k \leq \sum_{k=1}^{\infty} b_k$ since $\sum_{k=1}^{n} a_k \leq \sum_{k=1}^{n} b_k$.

Ex: $\sum_{K=1}^{\infty} \frac{1}{K^2+5}$ converges since $\frac{1}{K^2+5} \leq \frac{1}{K^2}$

and $\sum_{K=1}^{\infty} 1/K^2$ converges.

Ex: $\sum_{K=1}^{\infty} \frac{1}{K^2-5}$ Now this should converge since $1/(K^2-5)$ is "basically" $1/K^2$, but unfortunately $\frac{1}{K^2+5} \geq \frac{1}{K^2}$ so can't easily apply the comparison test.

Limit Comparison Test: $a_K, b_K > 0$. Suppose

$$\lim_{K\to\infty} \frac{a_K}{b_K} = \underset{\text{finite}}{L} > 0.$$

Then $\sum_{K=1}^{\infty} a_K$, $\sum_{K=1}^{\infty} b_K$ both converge or both diverge.

Ex: $\sum_{K=1}^{\infty} \frac{1}{K^2-5}$ Now

$$\lim_{K\to\infty} \frac{\frac{1}{K^2-5}}{\frac{1}{K^2}} = \lim_{K\to\infty} \frac{K^2}{K^2-5}$$

$$= 1$$

and so $\sum_{K=1}^{\infty} 1/(K^2-5)$ converges since $\sum_{K=1}^{\infty} 1/K^2$ does