Lecture 26: Applications of Taylor Series (§8.7/§8.8)

HW #9 (Nov 5): §8.7 #22, 25, 32, 35, 41
§8.8. #7, 10, 12, 15

Next time: More on §8.8, §8.9

[The story so far...]

Ex: Compute $\lim_{x \to 0} \dfrac{\sin x^3 - x^3}{x^9}$ [Note both top and bottom go to 0 as $x \to 0$]

Now for any x, we have

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$$

and so

$$\sin x^3 = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (x^3)^{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{6k+3}$$

$$= x^3 - \frac{1}{3!}x^9 + \frac{1}{5!}x^{15} - \frac{1}{7!}x^{21} + \cdots$$

Thus

$$\frac{\sin x^3 - x^3}{x^9} = \frac{\left(x^3 - \frac{1}{3!}x^9 + \frac{1}{5!}x^{15} + \cdots\right) - x^3}{x^9}$$

$$= \underbrace{-\frac{1}{3!} + \frac{1}{5!}x^6 - \frac{1}{7!}x^{12} + \cdots}$$

As $x \to 0$, all these terms go to 0 as well. Thus

$$\lim_{x\to 0} \frac{\sin x^3 - x^3}{x^9} = -\frac{1}{3!} = -\frac{1}{6}$$

[Can also do using L'Hopital's Rule three times.]

Ex: $\displaystyle \lim_{x\to 0} \frac{\sin x^3 - x^3}{\cos x^3 - 1}$

Now $\displaystyle \cos x^3 = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{6k}$

$$= 1 - \frac{1}{2}x^6 + \frac{1}{4!}x^{12} - \frac{1}{6!}x^{18} + \cdots$$

So

$$\frac{\sin x^3 - x^3}{\cos x^3 - 1} = \frac{\left(-\frac{1}{3!}x^9 + \frac{1}{5!}x^{15} - \cdots\right)\left(\frac{1}{x^6}\right)}{\left(-\frac{1}{2}x^6 + \frac{1}{4!}x^{12} - \frac{1}{6!}x^{18} + \cdots\right)\left(\frac{1}{x^6}\right)}$$

$$= \frac{-\frac{1}{3!}x^3 + \frac{1}{5!}x^9 - \cdots}{-\frac{1}{2} + \frac{1}{4!}x^6 - \frac{1}{6!}x^{12} + \cdots} \longrightarrow \frac{0}{-\frac{1}{2}} = 0$$

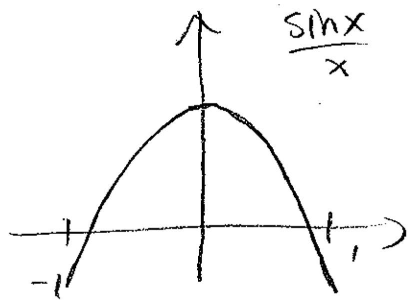
as $x \to 0$.

Ex: Estimate $\int_{-1}^{1} \frac{\sin x}{x}\,dx$. ← don't know antiderivative of.

Note: Integrand makes sense as $\lim_{x \to 0} \frac{\sin x}{x} = 1$,

as $$\frac{\sin x}{x} = \frac{x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots}{x} = 1 - \frac{1}{3!}x^2 + \frac{1}{5!}x^4 - \cdots$$

Now



$$\int_{-1}^{1} \frac{\sin x}{x}\,dx \overset{\text{approximately}}{\approx} \int_{-1}^{1} 1 - \frac{1}{3!}x^2 + \frac{1}{5!}x^4\,dx$$

$$= x - \frac{x^3}{18} + \frac{x^5}{600}\Big|_{x=-1}^{x=1} = 2\left(1 - \frac{1}{18} + \frac{1}{600}\right) = \frac{1703}{900} \approx 1.89222$$

[Note: Using that the series for $\frac{\sin x}{x}$ is alternating we could get precise error bounds and see our answer is correct to within $\frac{2}{7!} = 0.0003968\ldots$]

Remark: Sometimes even "extreme" approximations like $\sin x \approx x$ (for small x) are good enough...