Lecture 25: Taylor Series (§8.7)

HW #8 (Oct 29): §8.7: #15

Next time: Finish §8.7, start 8.8.

Last time: f infinitely differentiable at c.

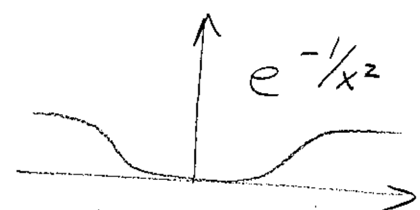
Taylor Series: $$\sum_{k=0}^{\infty} \frac{f^{(k)}}{k!}(x-c)^k$$

Q1: What is the radius of convergence r of ?

If $r > 0$ then $g(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}}{k!}(x-c)^k$ gives a function on $(c-r, c+r)$ with $g^{(k)}(c) = f^{(k)}(c)$ for all k.

Q2: Does $f(x) = g(x)$ on $(c-r, c+r)$?

[Gave an example where answers are (yes, no):



When the answer to Q2 is yes, the Taylor Polynomials

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}}{k!} x^k$$

give approximations to $f(x)$.

[In practice one usually works with these.]

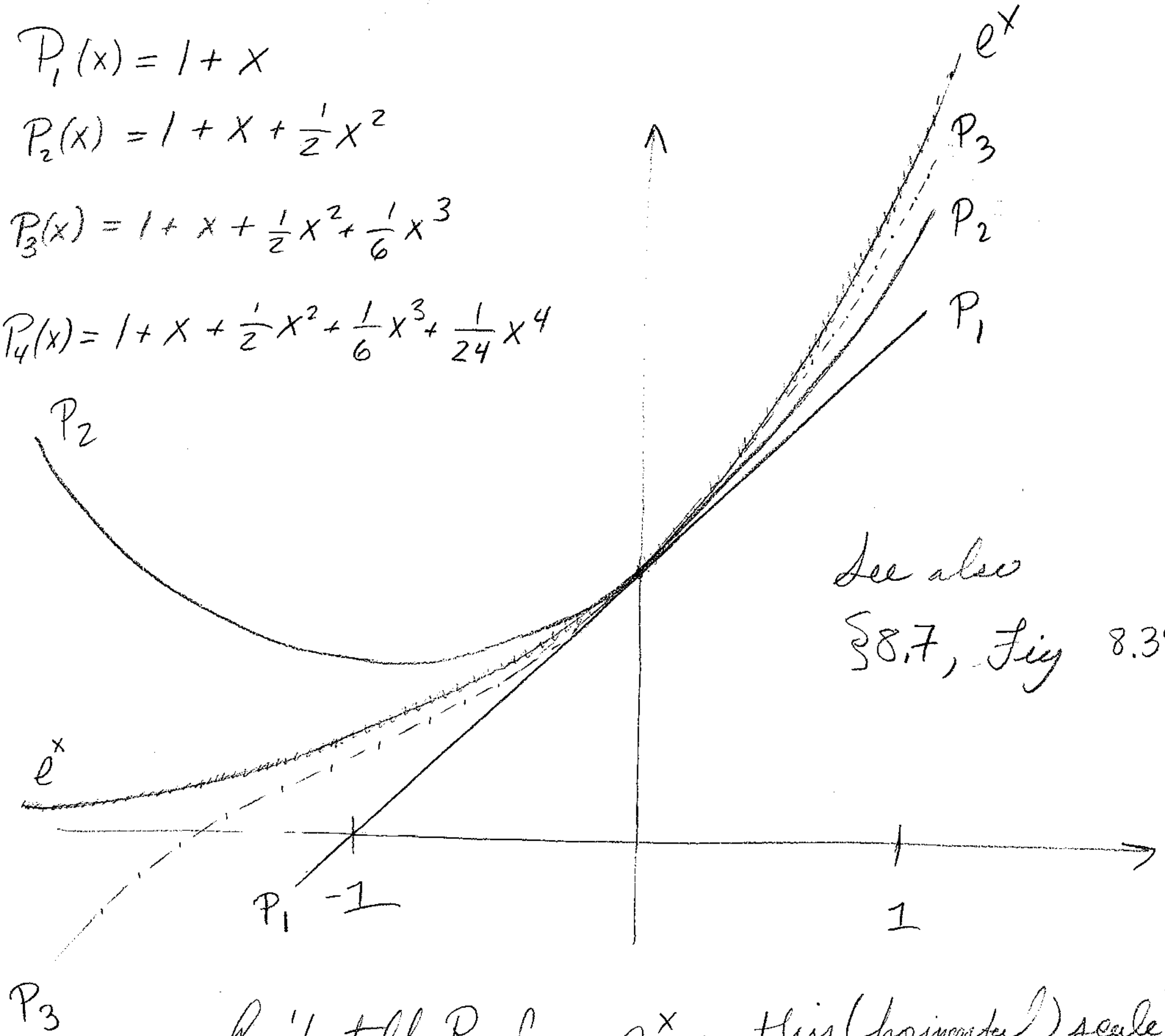
Ex: $f(x) = e^x$ Taylor series about 0: $\sum_{k=0}^{\infty} \frac{1}{k!} x^k$ rad of conv $= \infty$

$P_1(x) = 1 + x$

$P_2(x) = 1 + x + \frac{1}{2}x^2$

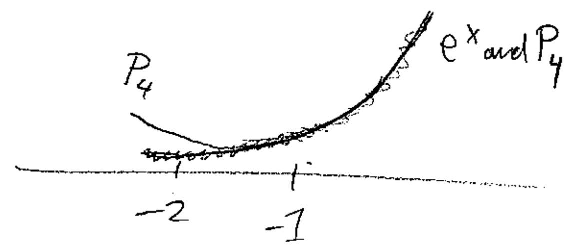
$P_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$

$P_4(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$



See also §8.7, Fig 8.39

Can't tell P_4 from e^x on this (horizontal) scale except at the very end



Note: Each P_n only approximates e^x "near" 0.

$$0 \leq R_1\left(\tfrac{1}{2}\right) \leq \frac{e^z}{8} \leq \frac{e^{1/2}}{8} \leq \frac{4^{1/2}}{8} = \frac{1}{4}$$

as $e \leq 4$

Thus $e^{1/2}$ is in $\left[3/2,\ 3/2 + \frac{1}{4}\right] = [1.5,\ 1.75]$.

For each x, the series $\sum_{k=0}^{\infty} \frac{1}{k!} x^k$ converges to e^x.

Reason:

Fix x. There is an M so that $|e^z| \leq M$ for all z between 0 and x [since e^x is continuous]

Then

$$|P_n(x) - e^x| = \left|\frac{e^z}{(n+1)!} x^{n+1}\right| \leq \frac{M}{(n+1)!} |x|^{n+1}$$

which goes to 0 as $n \longrightarrow \infty$. So $\sum_{k=0}^{\infty} \frac{1}{k!} x^k = e^x$

Ex: $e = \sum_{k=0}^{\infty} \frac{1}{k!}$: use this to estimate e to 10^{-10}

Note: $R_n(1) = \frac{e^z}{(n+1)!}$ for $0 \leq z \leq 1$. So $|R_n(1)| \leq \frac{3}{(n+1)!}$

In particular, if $n = 13$, then $|R_n(1)| < 3.4 \times 10^{-11}$

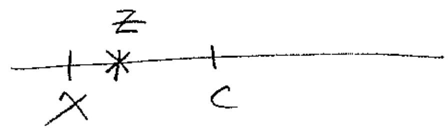
So $e \approx 2.7182818284\ldots$

<u>Taylor's Theorem</u>: Suppose f has $(n+1)$ derivatives on $(c-r, c+r)$ for some $r>0$.

Consider $P_n = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$. Then for x in $(c-r, c+r)$ the error

$$R_n(x) = f(x) - P_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$$

where z is between x and c.

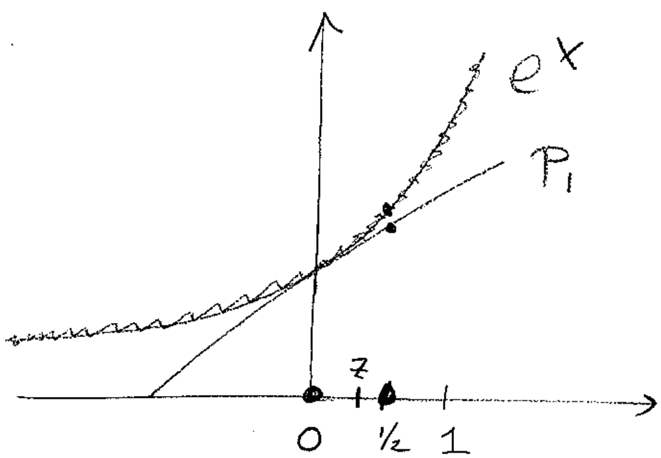


<u>Ex</u>: $f(x) = e^x \qquad P_1 = 1 + x$

At $x = 1/2$

$P_1(1/2) = 3/2 = 1.5$

$f(1/2) = e^{1/2} = 1.6487\ldots$



So Taylor's Theorem says:

$$R_1(1/2) = f(1/2) - P_1(1/2) = 0.1487\ldots = \frac{f^{(2)}(z)}{2}\left(\frac{1}{2}\right)^2 = \frac{e^z}{8}$$

for <u>some</u> z between 0 and $1/2$. (Turns out $z = 0.17376\ldots$)

Why useful? Well, e^x is increasing, so even <u>without</u> knowing what z is we get

Brownian Motion: Dust moving in sunlight

Brown (19th century): pollen grains on surface of water. Picked up by Einstein (1905)

2000 years earlier, the roman Lucretius used this as an argument for the existence of molecules.