Lecture 24: Taylor Series (§8.7)

HW #8 (Oct 29): §8.7: # 1, 2, 5, 12, 13

Next time: More on §8.7

Q: Given a function, say e^x or $\sin x$ or $\dfrac{\cos x + \ln x}{x^2 - x + 7}$, can we express it as a power series $\sum_{k=0}^{\infty} b_k (x-c)^k$

E.g. $\tan^{-1} x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots$ for x in $(-1,1)$

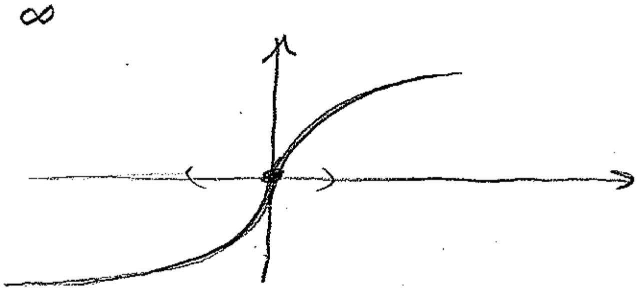
[Discuss usefulness of such things.]

Recall: $f(x) = \sum_{k=0}^{\infty} b_k (x-c)^k$ with rad of conv $r > 0$.

then $f'(x) = \sum_{k=0}^{\infty} k\, b_k (x-c)^{k-1}$ with same rad of conv.

Note: As a result, some fns don't have power series expansions. Eg can't express $x^{\frac{1}{3}}$ as $\sum_{k=0}^{\infty} b_k x^k$ since $\frac{d}{dx} x^{\frac{1}{3}} \Big|_{x=0} = \frac{1}{3} x^{-\frac{2}{3}} \Big|_{x=0} = \infty$

does have such expansions centered at other points, e.g. 1.



Consider $f(x) = \sum_{k=0}^{\infty} b_k (x-c)^k$. Then

$$f(x) = b_0 + b_1(x-c) + b_2(x-c)^2 + b_3(x-c)^3 + b_4(x-c)^4 + \cdots$$

$$f'(x) = 0 + b_1 + 2b_2(x-c) + 3b_3(x-c)^2 + 4b_4(x-c)^3 + \cdots$$

$$f''(x) = 0 + 0 + 2b_2 + 6b_3(x-c) + 12b_4(x-c)^2 + \cdots$$

$$f^{(3)}(x) = 0 + 0 + 0 + 6b_3 + 24b_4(x-c) + \cdots$$

$$f^{(4)}(x) = 0 + 0 + 0 + 0 + 24b_4 + \cdots$$

Notice:

$$\left.\begin{aligned} f(c) &= b_0 \\ f'(c) &= b_1 \\ f''(c) &= 2b_2 \\ f^{(3)}(c) &= 6b_3 \\ f^{(4)}(c) &= 24b_4 \end{aligned}\right\}$$

general case:

$$f^{(k)}(c) = (k!)\, b_4$$

Now suppose we're given f which is infinitely differentiable at c. The Taylor Series for f centered at c is:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

Q: ⓐ Does the series converge?

ⓑ If so, does it converge to $f(x)$?

[It depends! In many cases the answer is yes.]

Ex: $f(x) = e^x$. Then $f'(x) = e^x$, $f''(x) = e^x$, and $f^{(k)}(x) = e^x$

So Taylor about 0 is $\displaystyle\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$

For (a) we just apply the Ratio Test:

$$\lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \to \infty} \left| \frac{\frac{x^{k+1}}{(k+1)!}}{\frac{x^k}{k!}} \right| = \lim_{k \to \infty} \frac{|x|}{k+1} = 0 < 1$$

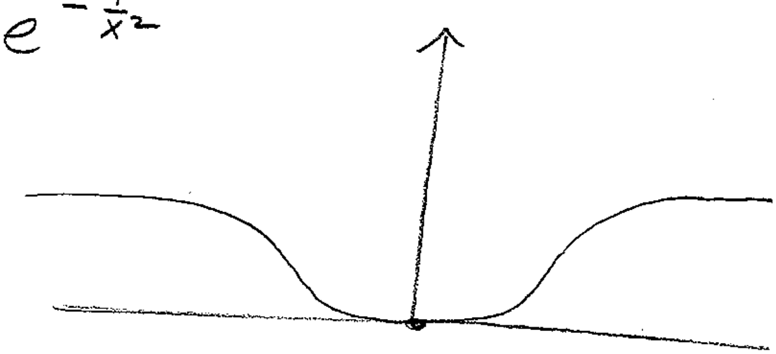
So converges for all x.

What about ⓑ? Aside: Here's what we want to avoid: $f(x) = e^{-\frac{1}{x^2}}$

Turns out

$f^{(k)}(0) = 0$

for all k. So the



Taylor series for f at 0 is

$$\sum_{k=0}^{\infty} \frac{0}{k!} x^k = 0$$ which converges, but not to f!