Lecture 23: Differentiating and integrating power series.

HW #8 (Oct 29) §8.6 # 2, 5, 6, 34, 35, 37

Next time: §8.7.

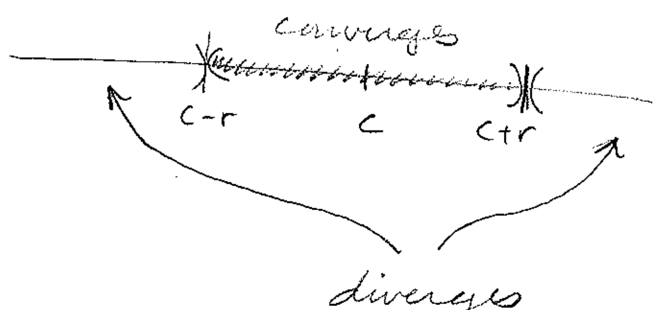
Last time: $\sum_{k=0}^{\infty} b_k (x-c)^k$

Three possibilities:

i) Series converges absolutely for all x.

ii) Series converges only for $x = c$.

iii) There is an r so that the series converges absolutely for x in $(c-r, c+r)$, diverges for x in $(-\infty, x-c)$ and $(x+c, \infty)$.



Ex: i) $\sum_{k=0}^{\infty} \frac{1}{k!} x^k$

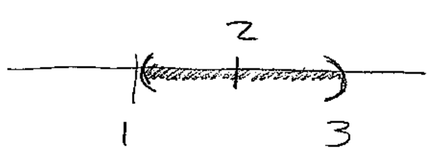
ii) $\sum_{k=0}^{\infty} k!(x-3)^k$ since

$$\lim_{k\to\infty} \left| \frac{(k+1)!\,(x-3)^{k+1}}{k!\,(x-3)} \right| = \lim_{k\to\infty} (k+1)|x-3|$$

$$= \begin{cases} \infty & \text{if } x \neq 3 \\ 0 & \text{if } x = 3 \end{cases}$$

iii) $\sum_{k=0}^{\infty} (-1)^{k+1} (x-2)^k$

Converges on $(1, 3)$, radius of convergence $= 1$.



Differentiating Power Series:

$$f(x) = \sum_{k=0}^{\infty} b_k (x-c)^k \quad \text{with radius of convergence } r > 0.$$

Fact: f is continuous and differentiable on $(c-r, c+r)$ with

$$f'(x) = \frac{d}{dx} f(x) = \frac{d}{dx}\left(b_0 + b_1(x-c) + b_2(x-c)^2 + b_3(x-c)^3 + \cdots\right)$$

$$= b_1 + 2b_2(x-c) + 3b_3(x-c)^2 + \cdots = \underbrace{\sum_{k=0}^{\infty} k b_k (x-c)^{k-1}}_{\text{abs. conv. on } (c-r,c+r)}$$

Ex: $f(x) = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + x^4 + \cdots$ abs. conv on $(-1, 1)$

$$f'(x) = \sum_{k=0}^{\infty} k x^{k-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

Now $f(x) = \frac{1}{1-x}$ so $f'(x) = \frac{+1}{(1-x)^2}$. Thus

$$4 = f'(1/2) = \sum_{k=0}^{\infty} k\left(\tfrac{1}{2}\right)^{k-1} = 1 + 2\tfrac{1}{2} + 3\tfrac{1}{4} + 4\tfrac{1}{8} + 5\tfrac{1}{16} + \cdots$$

or 1.

So

$$\sum_{k=1}^{\infty} k\left(\tfrac{1}{2}\right)^k = \sum_{k=1}^{\infty} k\,\tfrac{1}{2}\left(\tfrac{1}{2}\right)^{k-1} = \tfrac{1}{2}\sum_{k=1}^{\infty} k\left(\tfrac{1}{2}\right)^{k-1} = 2$$

[Much easier than on Honors HW!]

<u>Integration</u>: $f(x) = \sum_{k=0}^{\infty} b_k (x-c)^k$

Then

$$\int f(x)\,dx = \int \left(b_0 + b_1(x-c) + b_2(x-c)^2 + b_3(x-c)^3 + \cdots \right) dx$$

$$= \left(b_0 x + \frac{b_1}{2}(x-c)^2 + \frac{b_2}{3}(x-c)^3 + \frac{b_3}{4}(x-c)^4 + \cdots \right) + C$$

$$= \left(\underbrace{\sum_{k=0}^{\infty} \frac{b_k}{k+1}(x-c)^{k+1}}_{} \right) + C.$$

has same radius of convergence as original series.

<u>Application</u>: $f(x) = \sum_{k=0}^{\infty} (-1)^k x^k = \sum_{k=0}^{\infty} (-x)^k = \frac{1}{1+x}$

has radius of conv $= 1$

Now

$$\frac{1}{1+x^2} = f(x^2) = \sum_{k=0}^{\infty} (-1)^k x^{2k}$$

and

$$\tan^{-1} x = \int \frac{1}{1+x^2}\,dx = \int \sum_{k=0}^{\infty} (-1) x^{2k}\,dx$$

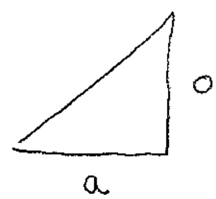
$$= \underbrace{\left(\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}\right)} + C$$

$$= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 - \cdots$$

Plug in $x = 0$, we get $\tan^{-1} 0 = 0 + C \implies C = 0$.

So

$$\tan^{-1} x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}$$



$\tan = \frac{o}{a}$

and

$\tan^{-1} x \in$

$$\frac{\pi}{4} = \tan^{-1} 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$$

an alternating series for π!

Taylor Series: (§8.7)

Given a function, like $\sin x$ or e^x or $\frac{x^2+1}{x+\ln x}$ how can we express it as a power series?

[Explain importance of doing so for transcendental functions, and even non-transcendental ones]

Consider $f(x) = \sum_{k=0}^{\infty} b_k (x-c)^k = b_0 + b_1(x-c) + b_2(x-c)^2 + \cdots$

Then

$$f(c) = b_0$$

$$f'(c) = b_1 \qquad f'(x) = \sum_{k=0}^{\infty} k\, b_k (x-c)^{k-1} = b_1 + 2b_2(x-c) + \cdots$$

$$f''(c) = 2b_2 \qquad f''(x) = \sum_{k=0}^{\infty} k(k-1) b_k (x-c)^{k-2}$$

$$f'''(c) = 3!\, b_3 \qquad = 2b_2 + 6b_3(x-c) + 12 b_4 (x-c)^2 + \cdots$$

$$\vdots$$

$$f^{(n)}(c) = n!\, b_n \qquad f^{(n)}(x) = \sum_{k=0}^{\infty} k(k-1)(k-2)\cdots(k-n+1)\, b_k (x-c)^k$$

$$= (n!\, b_n) + \cdots\cdots$$

Now suppose f is a given function, infinitely differentiable, then if f has a power series expansion about c, it must be

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!} x^k \qquad \text{Taylor Series}$$

Questions:
- Does this converge?
- If it converges, does it give f?