Lecture 22: Power series (§8.6)

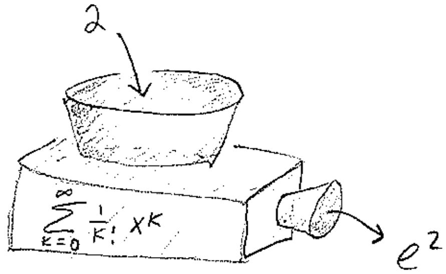
HW #8 (Oct 29): §8.6: 9, 12, 13, 15, 17, 25, 26

Next time: Taylor Series §8.7

Power series: $\displaystyle\sum_{k=0}^{\infty} \frac{1}{k!} x^k = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots = e^x$

Now

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots$$



converges for $|x| < 1$ and then we have

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}.$$

Gives approximations

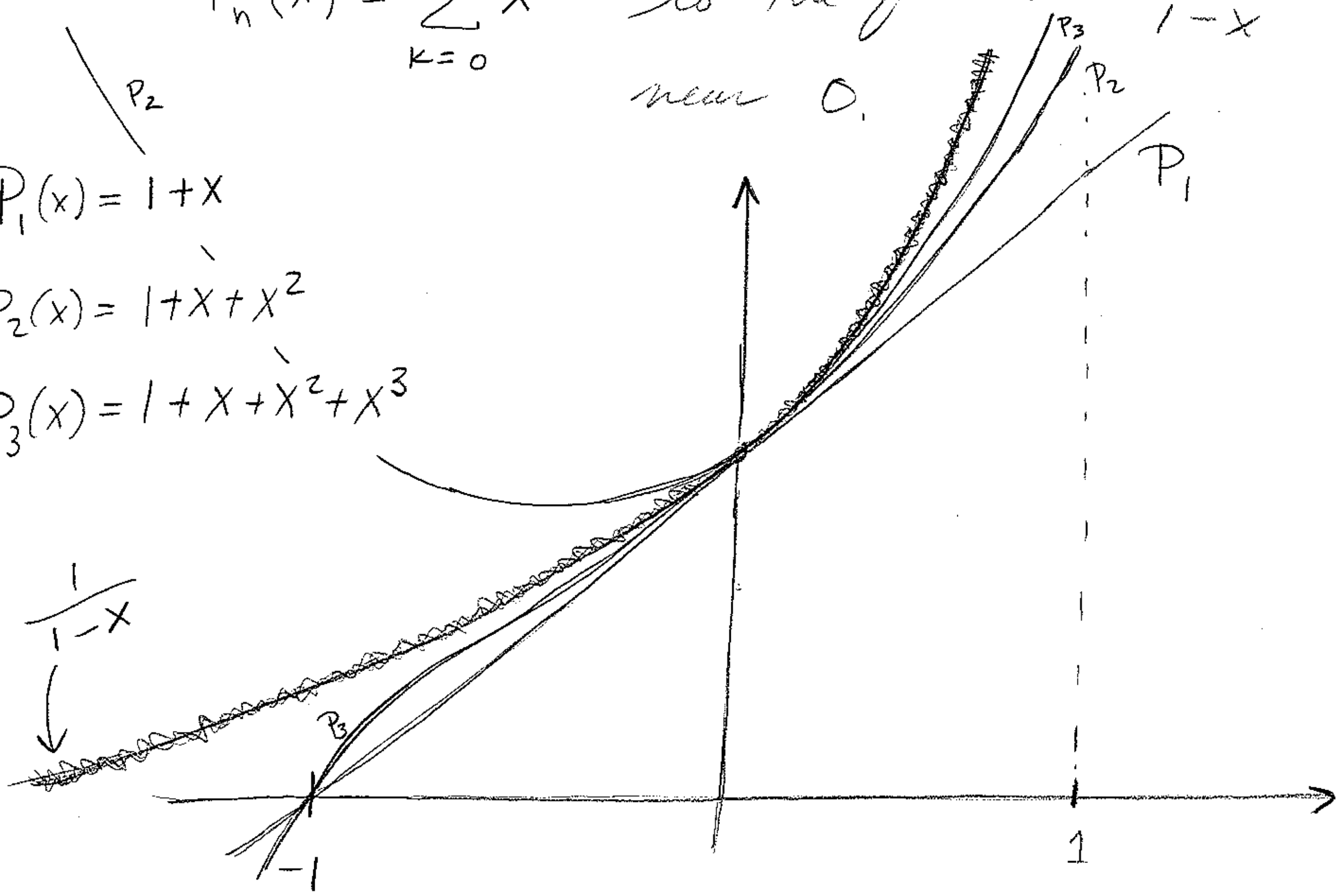
$$P_n(x) = \sum_{k=0}^{n} x^k$$

to the function $\frac{1}{1-x}$ near 0.

$P_1(x) = 1 + x$

$P_2(x) = 1 + x + x^2$

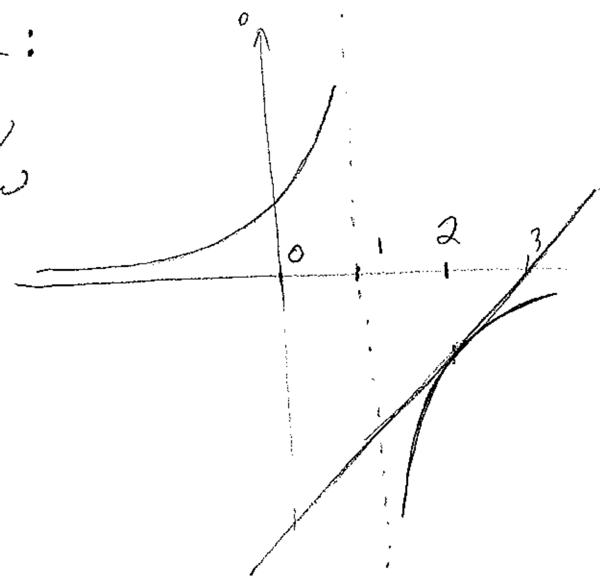
$P_3(x) = 1 + x + x^2 + x^3$



Notice that P_1 gives the tangent line to the graph of $\frac{1}{1-x}$ at 0: $\left.\frac{d}{dx}\frac{1}{1-x}\right|_{x=0} = \left.\frac{+1}{(1-x)^2}\right|_{x=0} = 1.$

P_2 is the best quadratic approximation...

Note series only converges for $|x| < 1$. There are other series that work elsewhere:



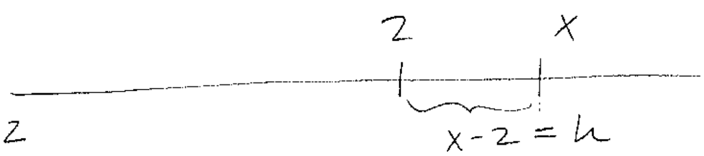
For instance, focus on $x = 2$. Now

$$\left.\frac{d}{dx}\left(\frac{1}{1-x}\right)\right|_{x=2} = \left.\frac{1}{(1-x)^2}\right|_{x=2} = 1$$

and so the tangent line is

$$y = -1 + (x-2) = x - 3$$

Next approximation turns out to be



$$P_2(x) = -1 + (x-2) - (x-2)^2$$

and in fact

$$\sum_{k=0}^{\infty} (-1)^{k+1}(x-2)^k = \frac{1}{1-x}$$ for x in $(1,3)$, that is when $|x-2| < 1$

since

$$\sum_{k=0}^{\infty}(-1)^{k+1}(x-2)^k = \sum_{k=0}^{\infty} -(-1)^k(x-2)^k$$

$$= -\sum_{k=0}^{\infty}(2-x)^k = \frac{-1}{1-(2-x)} = \frac{1}{1-x}$$

<u>General Power Series:</u> c, b_k fixed numbers

$$\sum_{k=0}^{\infty} \underbrace{b_k}_{\text{coefficients}} (x-c)^k$$

For which x does this make sense?

<u>Ex:</u> $\displaystyle\sum_{k=0}^{\infty} \frac{k}{2^k} x^k = \frac{1}{2}x + \frac{1}{2}x^2 + \frac{3}{8}x^3 + \cdots$

<u>Ratio Test:</u> [Generally works for power series.]

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{x\to\infty}\left|\left(\frac{(k+1)x^{k+1}}{2^{k+1}}\right)\cdot\left(\frac{2^k}{k\,x^k}\right)\right|$$

$$= \lim_{k\to\infty}\frac{(k+1)|x|}{2k} = \frac{|x|}{2}$$

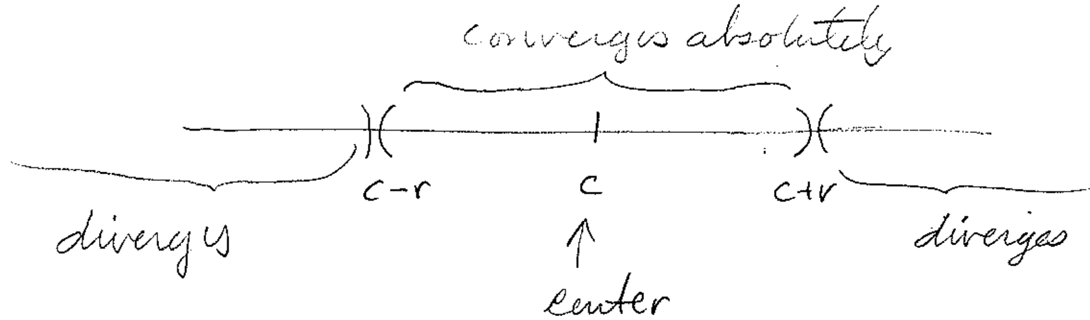
So, if $|x| < 2$, then $\frac{|x|}{2} < 1$ and so series is absolutely convergent. If $|x| > 2$, then it diverges. If $x = \pm 2$, then get $\sum_{k=1}^{\infty} k$ or $\sum_{k=1}^{\infty} (-1) k$ which diverge by the k^{th}-term test. So, the series converges for x in $(-2, 2)$.

For $\sum_{k=0}^{\infty} b_k (x-c)^k$ either

i) Converges absolutely for all x

ii) Converges only for $x = c$.

iii) For a certain $r > 0$ the series converges absolutely for x in $(c-r, c+r)$, diverges for x in $(-\infty, c-r)$ and $(c+r, \infty)$



Note: Doesn't tell us what happens at $c-r$ and $c+r$.

r called the <u>radius of convergence</u>.

Ex: i) $\sum_{k=0}^{\infty} \frac{1}{k!} x^k$ has infinite radius of convergence.

ii) $\sum_{k=0}^{\infty} k!\,(x-3)^k$ Ratio Test: $\frac{a_{k+1}}{a_k} = \frac{(k+1)!\,(x-3)^{k+1}}{k!\ (x-3)}$

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} (k+1)|x-3| \ldots$$

which diverges to ∞ unless $x = 3$.

iii) $\sum_{k=1}^{\infty} \frac{x^k}{k 4^k}$

Ratio Test: $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{k\,|x|}{(k+1)4} = \frac{|x|}{4}$

So if $\frac{|x|}{4} < 1 \iff |x| < 4$ then abs. conv

if $\frac{|x|}{4} > 1 \iff |x| > 4$ then diverges.

Radius of conv $= 4$. At $x = 4$ diverges

$x = -4$ conditionally conv.