Lecture 21: Midterm Review

Exam Friday: Bring sheet Covers: 8.1–8.5 (not 7.1!)

Review problems: Ch 8 review: True-False: #1–11

Exercises: 1–52 except 15

Also 8.5 #1–38.

Extra office hours: Today: 2:30 – 5:00 Math dept tutoring: 7–9:30 in Illini Hall

Thurs: 9–10:30

Overview:

Sequences: $\left\{a_k = \frac{1}{k}\right\}_{k=1}^{\infty} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \ldots\right\}$

[Note: no ε's on this exam.] Limits, monotone seqs

Series: $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots$ = diverges

$\sum_{k=1}^{\infty} \frac{1}{2^k} = 1$ [Diff between a seq and a series.]

$\sum_{k=1}^{\infty} a_k$
- converges
  - absolutely convergent (means $\sum_{k=1}^{\infty} |a_k|$ converges)
  - conditionally convergent
- diverges

Basic Examples. [+ telescoping series]

$\sum_{k=0}^{\infty} r^k$ converges for $|r| < 1$, diverges for $|r| \geq 1$

$\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges for $p > 1$, diverges for $p \leq 1$

Tests that apply to any series: $\sum_{k=1}^{\infty} a_k$

k-term test: If $\lim_{k\to\infty} a_k \neq 0$ then $\sum_{k=1}^{\infty} a_k$ diverges

Ratio test: Suppose $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = L$

If $L < 1$, $\sum_{k=1}^{\infty} a_k$ is absolutely convergent

$L > 1$ diverges. $L = 1$ inconclusive

Tests for $\sum_{k=1}^{\infty} a_k$ with $a_k \geq 0$ for all k.
- Integral Test.
- Comparison Test.
- Limit Comp Test.

[Relate to abs. conv. Note that these tests always interrelate two things]

Limit Comp Test: $\sum_{k=1}^{\infty} a_k$ $\quad \sum_{k=1}^{\infty} b_k$

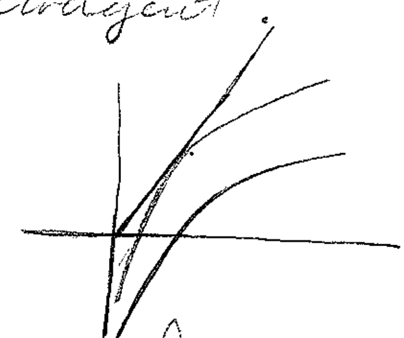
If $\lim_{k\to\infty} \frac{a_k}{b_k} = L$ not $0, \infty$

then the series both converge, or both diverge.

Alternating Series Test: $\sum_{k=1}^{\infty}(-1)^k a_k$ where $a_k > 0$.

Review #48:

Is $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{3}{(\ln k)+1}$ abs conv, cond conv, divergent?



Check abs conv: Does $\sum_{k=1}^{\infty}\frac{3}{(\ln k)+1}$ converge?

No: We know $\ln k \leq k$ for $k \geq 1$ and in fact $\ln k + 1 \leq k$ for k large. So $\frac{3}{k} \leq \frac{3}{(\ln k)+1}$ for large k.

As $\sum_{k=1}^{\infty}\frac{3}{k} = 3\sum_{k=1}^{\infty}\frac{1}{k}$ diverges, $\sum_{k=1}^{\infty}\frac{3}{(\ln k)+1}$ diverges by the Comp Test.

Does it converge? $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{3}{(\ln k)+1} = \sum_{k=1}^{\infty} (-1)^{k+1} a_k$

Yes by Alt. Series Test: where $a_k = \frac{3}{(\ln k)+1}$

1) a_k decreasing, i.e. $a_{k+1} \leq a_k$ for all k.

Check:

$$\frac{a_{k+1}}{a_k} = \frac{\frac{3}{\ln(k+1)+1}}{\frac{3}{\ln(k)+1}} = \frac{\ln(k)+1}{\ln(k+1)+1} < 1 \text{ since } \ln(k) < \ln(k+1)$$

2) $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{3}{\ln(k)+1} = 0$ ✓

Error Estimates: Estimate $\underbrace{\sum_{k=1}^{\infty} (-1)^{k+1} \frac{3}{(\ln k)+1}}_{=S}$ to within 0.1

Let $S_n = \sum_{k=1}^{n} (-1)^{k+1} \frac{3}{\ln(k)+1}$. How big does n have to be so then $|\underbrace{S - S_n}_{R_n}| < 0.1$ By A.S.T., $|S - S_n| \leq \underbrace{\frac{3}{\ln(n+1)+1}}_{a_{n+1}}$

Just pick n so that $\frac{3}{\ln(n+1)+1} < \frac{1}{10}$, that is

$$30 < \ln(n+1)+1 \iff 29 < \ln(n+1) \iff e^{29} < n+1$$

That is $n > e^{29} - 1 \approx 3.9 \times 10^{12}$

So just add up the first 4 trillion terms....

Ex: $\sum_{k=1}^{\infty} k \cdot \frac{(-3)^k}{(k!)^2}$: abs. conv, cond conv, or divergent?

Abs. conv by the ratio test:

$$\left| \frac{a_{k+1}}{a_k} \right| = \frac{(k+1)\frac{3^{k+1}}{((k+1)!)^2}}{k\frac{3^k}{(k!)^2}} = \frac{(k+1)\, 3^{k+1}\, (k!)^2}{k\, ((k+1)!)^2\, 3^k}$$

$$= \frac{(k+1)3}{k(k+1)^2} = \frac{3k+3}{k^3+2k^2+k} \quad \text{as} \quad \frac{k!}{(k+1)!} = \frac{1}{k+1}$$

Now

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \frac{3k+3}{k^3+2k^2+k} \frac{(1/k)}{(1/k)} = \lim_{k\to\infty} \frac{3+3/k}{k^2+2k+1}$$

$= 0$ As this is < 1, the Ratio test says the series is absolutely convergent.

Ex: $\sum_{k=1}^{\infty} \frac{k^2+k^{1/2}}{k^2+k^{5/2}}$ diverges by L.C.T with $\frac{1}{k^{1/2}}$

If time remains, do sequence questions, like with the squeeze theorem, or checking if a seq. is bounded.