Lecture 20: Conditionally Convergent Series (§8.5) / Power Series (§8.6)

Exam Friday: Bring sheet Covers: 8.1–8.5 (not 7.1!)

Review Problems:

Chapter 8 review: True-False: #1–11
Ex: 1–52 except 15.
Also 8.5 #1–38

Review Wed: Email me topics (nmd@illinois.edu)

Conditionally Convergent: $\sum_{k=1}^{\infty} a_k$ converges but $\sum_{k=1}^{\infty} |a_k|$ diverges

Ex: $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \cdots = \ln 2$

Fun Fact: The terms of a conditionally convergent series can be rearranged so they sum to any number, e.g. π or e or 17. Or rearranged so as to diverge.

[With an absolutely convergent series, the order of the terms doesn't matter. Will rearrange the alt. harmonic series so it diverges.]

Positive Terms: $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \cdots = \displaystyle\sum_{k=0}^{\infty} \frac{1}{2k+1}$

Diverges by the integral test (or L.C.T with $\frac{1}{k}$)

Break into pieces with sum ≥ 1:

$$1 + \underbrace{\left(\tfrac{1}{3} + \tfrac{1}{5} + \cdots + \tfrac{1}{29}\right)}_{1.0025} + \underbrace{\left(\tfrac{1}{31} + \cdots + \tfrac{1}{221}\right)}_{1.0006} + \underbrace{\left(\tfrac{1}{223} + \cdots + \tfrac{1}{1649}\right)}_{1.0029} + \left(\tfrac{1}{1651} + \cdots\right) +$$

Now insert a negative term between each block

$$\underbrace{1 - \tfrac{1}{2}}_{\geq 1/2} + \underbrace{\left(\tfrac{1}{3} + \cdots + \tfrac{1}{29}\right) - \tfrac{1}{4}}_{\geq 1/2} + \underbrace{\left(\tfrac{1}{31} + \cdots + \tfrac{1}{221}\right) - \tfrac{1}{6}}_{\geq 1/2} + \underbrace{\left(\tfrac{1}{223} \cdots + \tfrac{1}{1649}\right) - \tfrac{1}{8}}_{1/2} + \cdots$$

= diverges to ∞.

So we've rearranged the alt. harmonic series into a divergent series. [Can't do with an absolutely conv. series.]

Compare: $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \cdots$

where the pos terms sum to about 1.233

Power Series (§8.6)

$$\sum_{k=0}^{\infty} \frac{1}{k!} x^k = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots$$

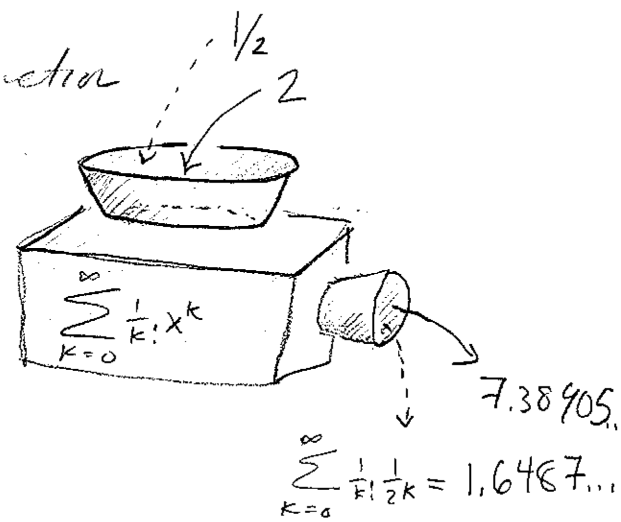
Here x is a variable for which we can plug in specific values. E.g. take $x=2$, and $\sum_{k=0}^{\infty} \frac{1}{k!} 2^k \approx 7.38905$

This particular power series converges for <u>any</u> fixed x by the ratio test:

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \frac{\frac{|x|^{k+1}}{(k+1)!}}{\frac{|x|^k}{k!}} = \lim_{k\to\infty} \frac{|x|}{k+1} = 0.$$

Thus, the power series defines a function

In fact $\boxed{\sum_{k=0}^{\infty} \frac{1}{k!} x^k = e^x}$



Many functions can be expressed this way

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{6!}x^6 - \frac{1}{8!}x^8 - \ldots$$

$$\ln x = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (x-1)^k = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 -$$

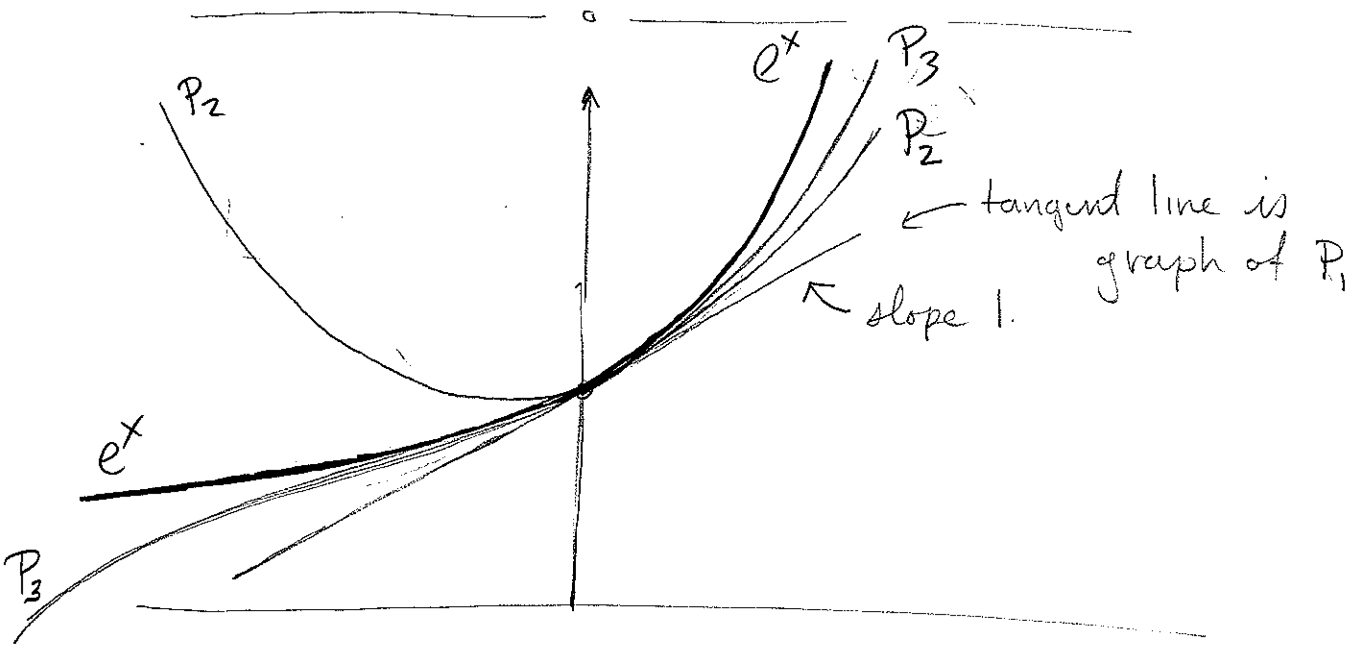
<u>Why?</u>

<u>Useful</u>: Find $\cos 1$ to within $0.0001 = \frac{1}{10{,}000}$

$$\cos 1 = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} = 1 - \frac{1}{2} + \frac{1}{24} - \frac{1}{720} + \frac{1}{40320} -$$

By alternatering series test, the error S_n and $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}$ is at most $|a_{n+1}|$. Thus $1 - \frac{1}{2} + \frac{1}{24} - \frac{1}{720} = \frac{389}{720} = 0.5402777\ldots$ is within $\frac{1}{40320}$ of $\cos 1 = 0.540302305\ldots$

Natural: Power series come out of looking at polynomial approximations of functions.



Equation of tangent line $\quad y = x+1 \quad$ as $\left(\frac{d}{dx}e^x\right)\Big|_{x=0} = \underline{1}$

$$P_1 = 1 + x$$

$$P_2 = 1 + x + \frac{1}{2}x^2 \qquad \text{"best fit parabola"}$$

$$P_3 = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 \qquad \text{"best fit cubic"}$$

Point: $\left(\frac{d}{dx^k} P_n(x)\right)\Big|_{x=0} = \frac{d}{dx^k} e^x\Big|_{x=0} = \underline{1}$