Lecture 19: Absolute convergence (§8.5)

HW #7: (Oct 15) §8.5: #5, 6, 11, 17, 18, 24, 27, 33, 36

Next time: §8.6

A series $\sum_{k=1}^{\infty} a_k$ is <u>absolutely</u> <u>convergent</u> if $\sum_{k=1}^{\infty} |a_k|$ converges.

<u>Key Fact</u>: If a series is absolutely convergent, then it converges.

<u>Ex</u>: $\sum_{k=1}^{\infty} \frac{\cos k}{k^2} = \cos 1 + \frac{1}{4}\cos 2 + \frac{1}{9}\cos 3 + \frac{1}{16}\cos 4 + \ldots$

Now $\sum_{k=1}^{\infty} \left|\frac{\cos k}{k^2}\right|$ converges by the Comparison Test

as $\left|\frac{\cos k}{k^2}\right| \le \frac{1}{k^2}$ and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges. Thus

$\sum_{k=1}^{\infty} \frac{\cos k}{k^2}$ converges absolutely, and hence converges.

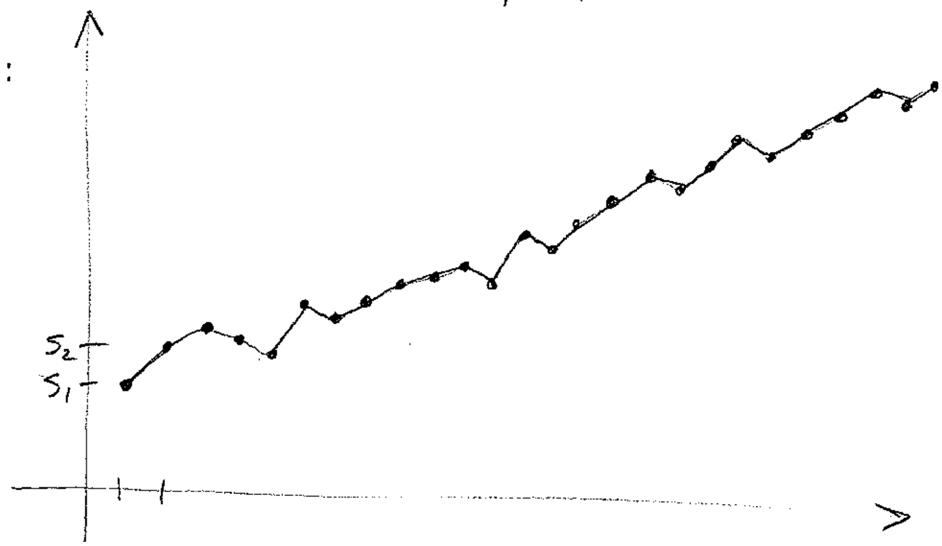
<u>Ex</u>: $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ converges, but is <u>not</u> absolutely convergent.

This is called <u>conditionally</u> <u>convergent</u>.

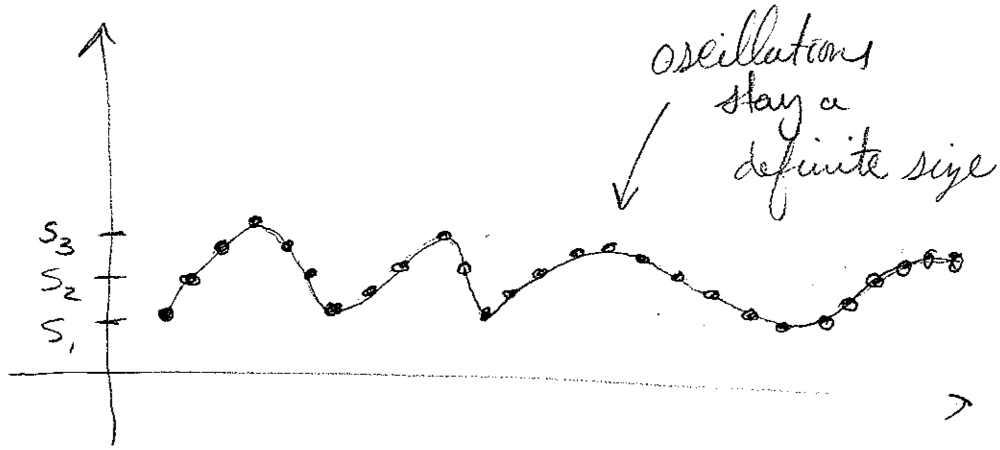
Idea: Two kinds of divergent series: $\sum_{k=1}^{\infty} a_k$

Partial sums go to ∞:

$$S_n = \sum_{k=1}^{n} a_k$$

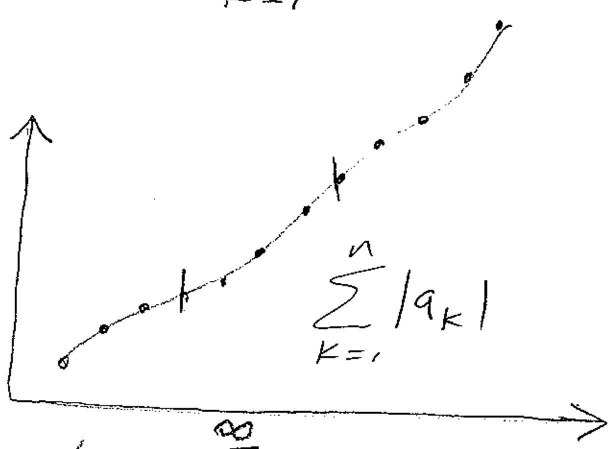


Oscillation:



In the first case, looking at $\sum_{k=1}^{n} |a_k|$ we see those partial sums grow even faster, so $\sum_{k=1}^{\infty} |a_k|$ diverges

In the second case, we get



and so $\sum_{k=1}^{\infty} |a_k|$ diverges.

Thus if $\sum_{k=1}^{\infty} a_k$ diverges, so does $\sum_{k=1}^{\infty} |a_k|$.

Ratio Test: [Last one!] $\sum_{k=1}^{\infty} a_k$ with $a_k \neq 0$.

Suppose
$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = L$$

If $L < 1$ then the series is absolutely convergent.
If $L > 1$ or $L = \infty$ the series diverges.
If $L = 1$ this test is sadly useless.

Point: If $L > 1$ then $|a_k|$ increases, so $\sum_{k=1}^{\infty} a_k$ diverges

If $L < 1$ and $L > 0$, then $|a_k|$ looks like CL^k and hence converges.

If $L = 0$, $|a_k| < \frac{1}{2^k}$ for large L, hence converges.

Ex: $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k e^k}{k!}$

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \frac{\frac{e^{k+1}}{(k+1)!}}{\frac{e^k}{k!}} = \lim_{k\to\infty} \frac{e}{k+1} = 0$$

So the ratio test says this is absolutely convergent.

Ex: $\sum_{k=1}^{\infty} \frac{1}{k}$ Ratio test fails as $\lim_{k\to\infty} \frac{\frac{1}{k+1}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{k}{k+1} = 1$

Ex: $\sum_{k=1}^{\infty} \frac{1}{k^2}$ Similar $\lim_{k\to\infty} \frac{\frac{1}{(k+1)^2}}{\frac{1}{k^2}} = \lim_{k\to\infty} \frac{k^2}{(k+1)^2} = 1$

[As one diverges and the other converges, you can see why we can't fix the limit test to handle $L = 1$]

[Also, the Root Test.]

Fun with conditionally convergent series:

The terms of $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ can be rearranged to sum to any number. This is not the case with absolutely convergent series, which always sum to the same number no matter what the order of the terms.