Lecture 15: Infinite Series (§8.2)

HW #6: Due Oct 8:

§8.2: 1, 7, 8, 16, 19, 20, 35

Next time: § 8.3

For a sequence $\{a_k\}_{k=1}^{\infty}$, we say $\sum_{k=1}^{\infty} a_k$ converges if when we look at the partial sums

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$

the limit $\lim_{n\to\infty} S_n$ exists. In this case $\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} S_n$.

Ex: $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ Now $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$

and so

$$S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(1/k - 1/k+1\right)$$

Telescoping sum

$$= (1 - \underbrace{1/2) + (1/2}_{\text{cancel}} - \underbrace{1/3) + (1/3}_{\text{cancel}} - \underbrace{1/4) + (1/4}_{\text{cancel}} - \underbrace{1/5) + \cdots}_{\text{cancel}} + \overbrace{(1/n}^{\text{cancel}} - 1/n+1)$$

So $S_n = 1 - \frac{1}{n+1}$ and $\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \lim_{n\to\infty} S_n = 1$.

<u>Geometric Series:</u> $\sum_{k=0}^{\infty} r^k = 1 + r + r^2 + r^3 + r^4 + \dots$ [r fixed]

[Compare: $\sum_{k=1}^{\infty} 1/2^k = 1/2 + 1/4 + 1/8 + \dots$]

Now

$$S_n = \sum_{k=0}^{n} r^k = 1 + r + r^2 + r^3 + \dots + r^n = \frac{1 - r^{n+1}}{1-r}$$

because

$$(1-r)S_n = (1-r)(1+r+r^2+\dots+r^n)$$

$$= (1+r+r^2+\dots+r^n) - (r+r^2+r^3+\dots+r^{n+1})$$

$$= (1 - r^{n+1})$$

So

$$\sum_{k=0}^{\infty} r^k = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1-r^{n+1}}{1-r} = \begin{cases} \frac{1}{1-r} & \text{if } |r| < 1 \\ \text{diverges} & \text{if } |r| \geq 1. \end{cases}$$

<u>Ex:</u> $\sum_{k=0}^{\infty} (1/5)^k = \frac{1}{1 - 1/5} = 5/4$

$\sum_{k=0}^{\infty} 5^k = 1 + 5 + 25 + 125 + \dots$ diverges

Ex: $\sum_{k=1}^{\infty} 1/2^k = \sum_{k=0}^{\infty} (1/2)^k - 1 = \frac{1}{1-1/2} - 1 = 1$

If $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{k\to\infty} a_k = 0$

Idea: $S_n = S_{n-1} + a_n$. So if the S_n converge, then must be changing by less and less as $n \to \infty$. In symbols: If $\sum_{k=1}^{\infty} a_k$ converges, say $\lim_{n\to\infty} S_n = L$, then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} S_n - S_{n-1} = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}$$

$$= L - L = 0.$$

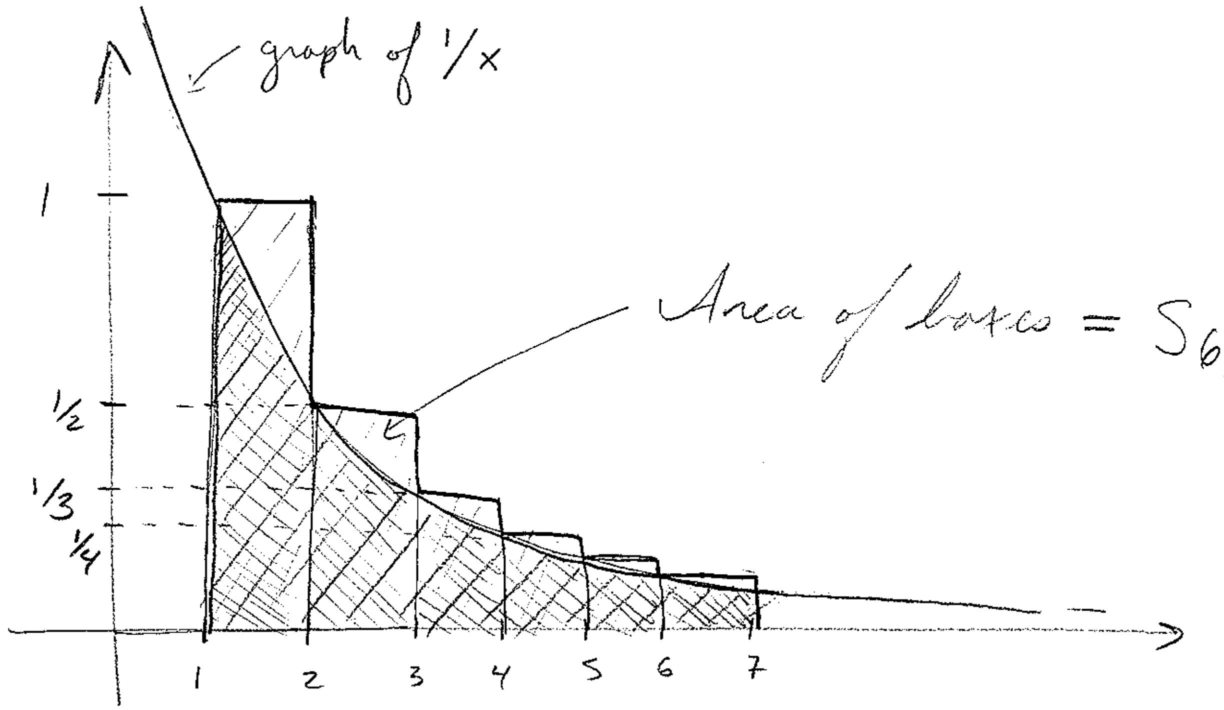
Ex: $\sum_{k=1}^{\infty} \frac{k+3}{k+2}$ diverges since $\lim_{k\to\infty} \frac{k+3}{k+2} = 1 \neq 0$.

Important Note: $\lim_{k\to\infty} a_k = 0$ does not mean that $\sum_{k=1}^{\infty} a_k$ converges!

Harmonic Series: $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k} = 1 + 1/2 + 1/3 + 1/4 + 1/5 + \cdots$

Diverges even though $\displaystyle\lim_{k\to\infty} \frac{1}{k} = 0$. Reason:

$$S_n = 1 + 1/2 + 1/3 + \cdots + 1/n$$



So

$$S_n = \text{area of boxes} \geq \int_1^{n+1} 1/x \, dx = \ln x \Big|_1^{n+1}$$

$$= \ln(n+1)$$

Hence $\displaystyle\lim_{n\to\infty} S_n$ diverges as $\ln(n+1) \longrightarrow \infty$ as $n \to \infty$