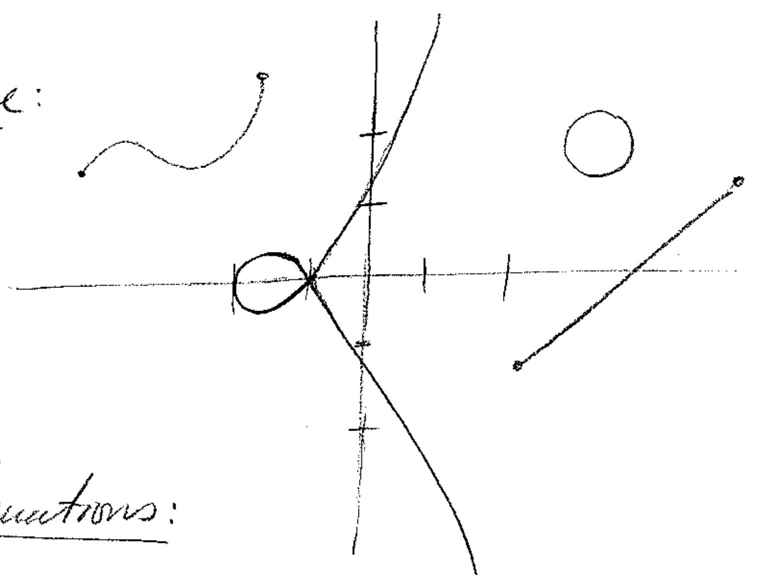
Lecture 29: Plane curves (§9.1 – 9.2)

HW (Nov 12): §9.1: 33, 36
§9.2: 2, 3, 12, 13

Reminder: Exam next Friday
Next time: More on §9.2

Plane curve:

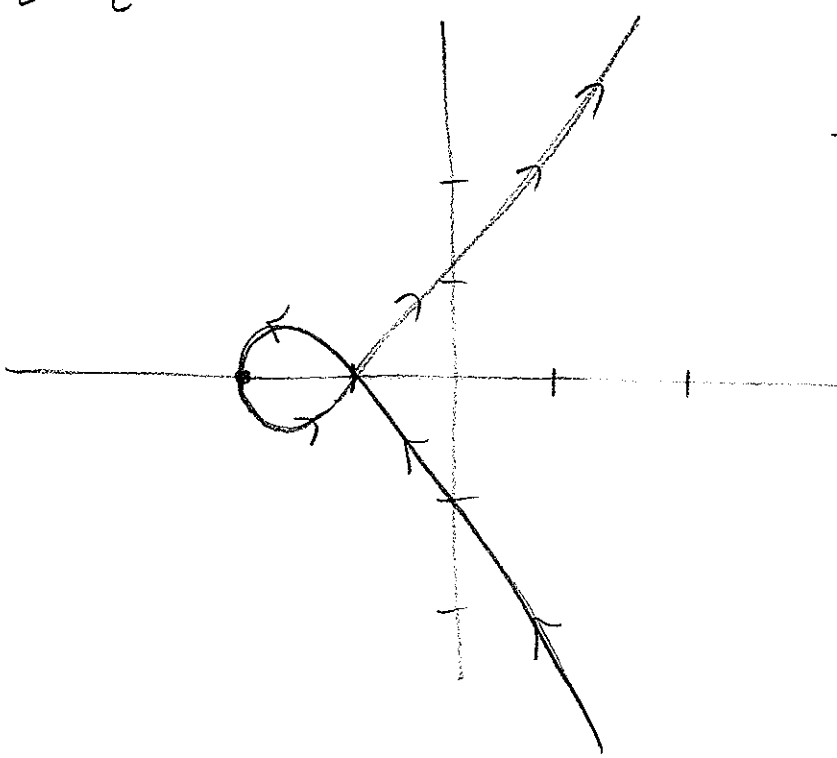


Parametric Equations:

$x(t) = t^2 - 2$
$y(t) = t^3 - t$

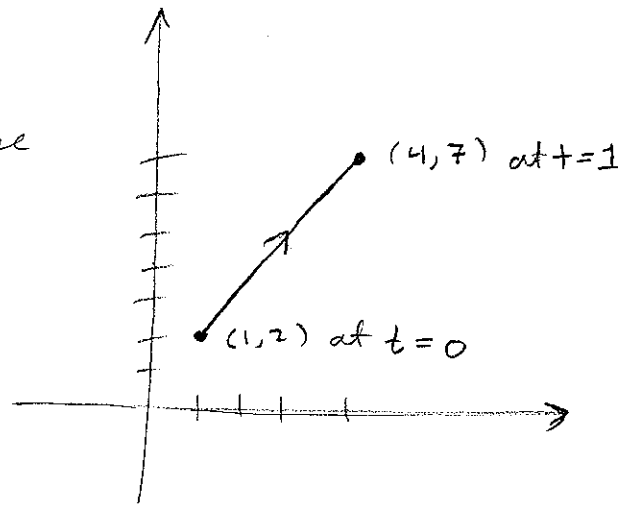
for t in $(-\infty, \infty)$

$t = 0 \quad (x, y) = (-2, 0)$
$t = 1 \quad (x, y) = (-1, 0)$
$t = -1 \quad (x, y) = (-1, 0)$



Ex: Find a parametric equation for the line segment joining (1,2) to (4,7).

Many ways to do this. A simple one would take the form



$$x = a + bt \qquad \text{for some } a, b, c, d.$$
$$y = c + dt$$

$\underline{t=0}$: $\quad x = 1 = a + b\cdot 0 = a \implies a = 1$

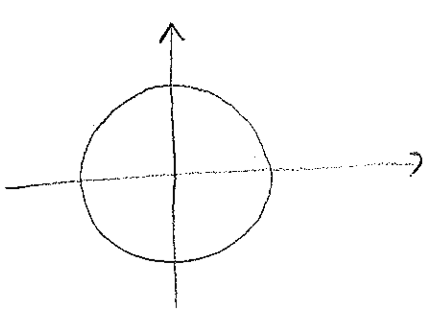
$\underline{t=1}$: $\quad x = 4 = a + b\cdot 1 = a+b \implies b = 3$

Similarly, $c=2$ and $d=5$. So our parametrization is:

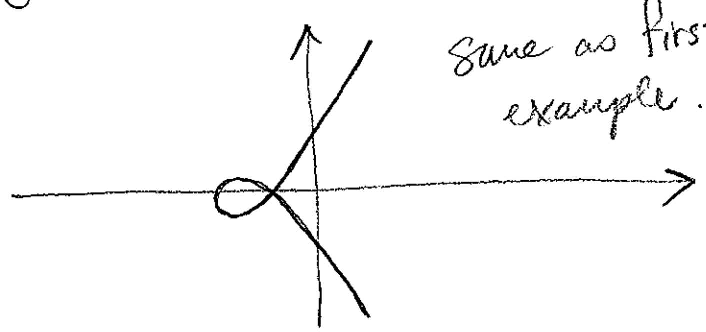
$$x = 1+3t \qquad y = 2+5t \qquad \text{for } 0 \le t \le 1$$

Plane curves can also be given by equations:

$x^2+y^2=4$



$y^2-(x+2)(x+1)^2=0$

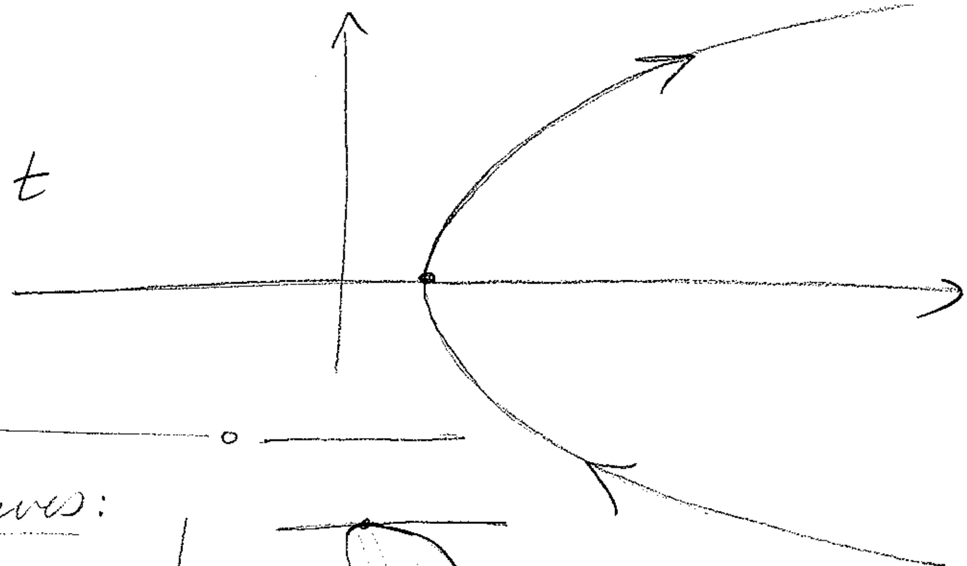


same as first example.

Ex: $x - y^2 + 1 = 0 \iff x = y^2 + 1$

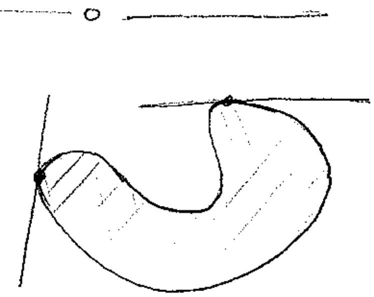
Parameterization:

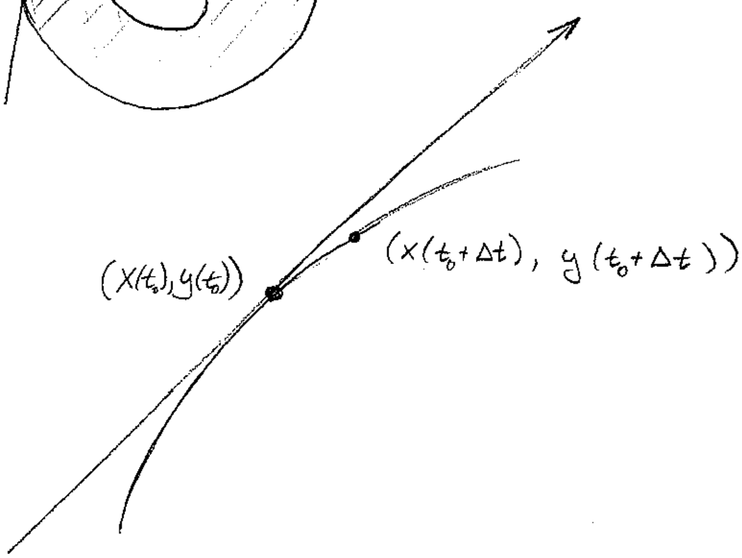
$x = t^2 + 1 \qquad y = t$



Properties of plane curves:

Tangent lines, areas
(Section 9.2)





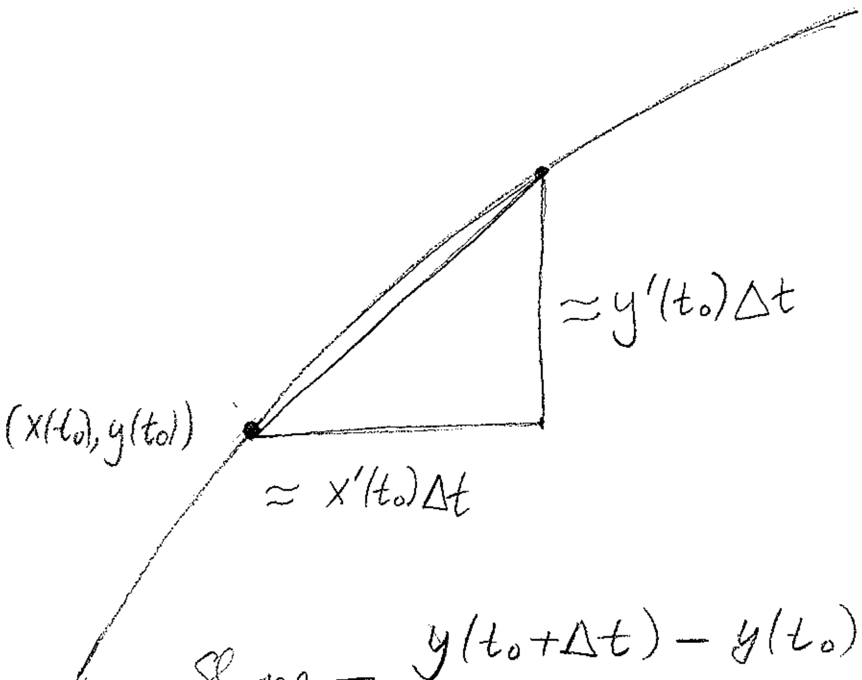
$x(t_0+\Delta t) \approx x(t_0) + x'(t_0)\Delta t$

Can think of this as
our Talyor approx of degree 1 centered at t_0:

$$x(t) \approx x(t_0) + x'(t_0)(t - t_0)$$

and now take $t = t_0 + \Delta t$.

So: $X(t_0+\Delta t) - X(t_0) \approx X'(t_0)\Delta t$ ← error should be like $C \cdot (\Delta t)^2$

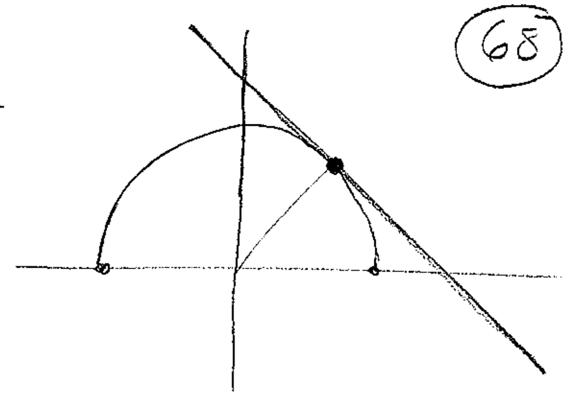


Slope of segment $= \dfrac{y(t_0+\Delta t) - y(t_0)}{X(t_0+\Delta t) - X(t_0)} \approx \dfrac{y'(t_0)}{X'(t_0)}$

Taking $\Delta t \to 0$, find slope of tangent line is

$$\frac{y'(t_0)}{X'(t_0)} \underset{\substack{\uparrow \\ \text{notation}}}{=} \frac{dy}{dx}(t_0) \underset{\substack{\uparrow \\ \text{good way} \\ \text{to remember}}}{=} \frac{dy}{dt}\frac{dt}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Ex: $x(t) = 2\cos t$ $\quad 0 \le t \le \pi$
$y(t) = 2\sin t$



$t_0 = \frac{\pi}{4}$ $\quad x = 2\cos\frac{\pi}{4} = \sqrt{2}$
$y = 2\sin \pi/4 = \sqrt{2}$

$x'(t) = -2\sin t$
$y'(t) = 2\cos t$

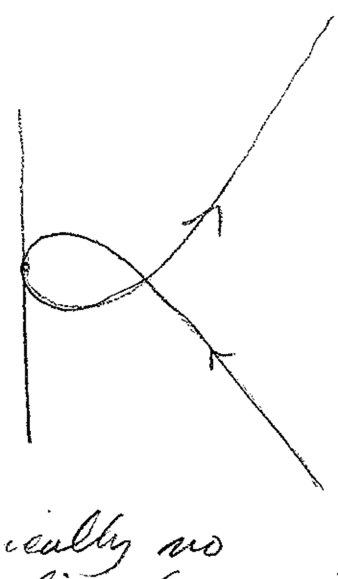
Slope of tangent line is

$$\frac{dy}{dx} = \frac{y'(\pi/4)}{x'(\pi/4)} = \frac{\sqrt{2}}{-\sqrt{2}} = -1$$

Ex: $x(t) = t^2 - 2$ $\quad x'(t) = 2t$
$y(t) = t^3 - t$ $\quad y'(t) = 3t^2 - 1$

Slope of tangent line at 0 is

$$\frac{dy}{dx} = \frac{y'(0)}{x'(0)} = \frac{-1}{0}$$



means the slope is vertical:

Note: If both $x'(t_0) = 0$
$y'(t_0) = 0$ then typically no tangent line. (corner).