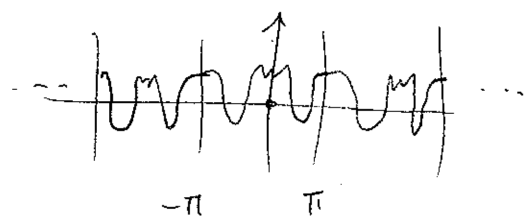
Lecture 27: Fourier Series (§8.9)

HW (Nov 5): §8.9: #1

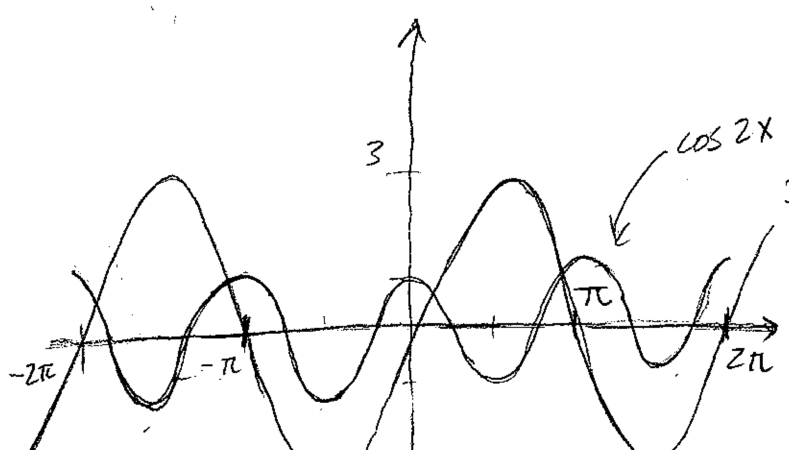
Next time: (§9.1)

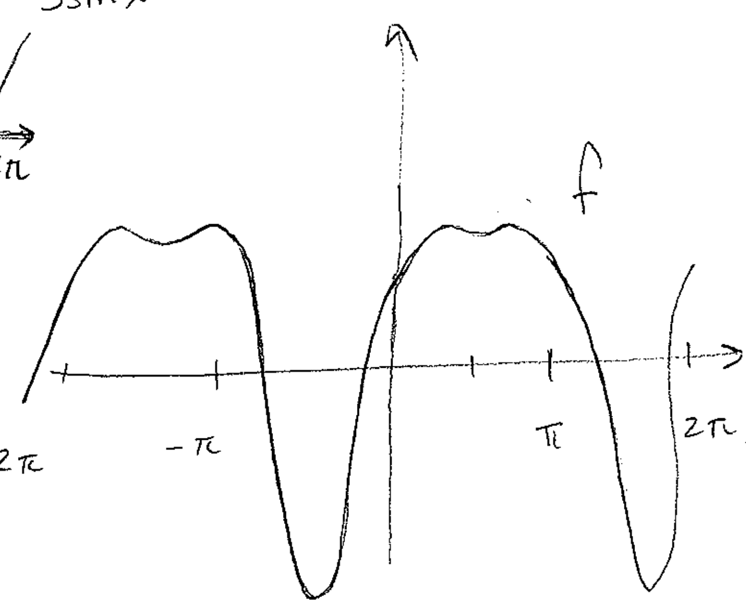
Fourier Series: $\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \sin kx + b_k \cos kx)$

used to describe functions that are periodic of period 2π: $f(x+2\pi) = f(x)$



Ex: $3 \sin x + \cos 2x = f(x)$





Put up before hand.

[Pulling apart a signal into its components;
overtones and harmonics; A = 440 Hz]

Q: Which functions have Fourier series, and how do we find the coefficients?

Suppose $f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$.

Then
$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} \frac{a_0}{2}\,dx + \sum_{k=1}^{\infty} \left(a_k \int_{-\pi}^{\pi} \cos kx\,dx + b_k \int_{-\pi}^{\pi} \sin kx\,dx \right)$$

$$= a_0 \pi + \sum_{k=1}^{\infty} (0+0) = a_0 \pi.$$

So
$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

Also
$$\int_{-\pi}^{\pi} f(x) \cos nx\,dx = \int_{-\pi}^{\pi} \frac{a_0}{2} \cos nx\,dx + \sum_{k=1}^{\infty} \left(a_k \int_{-\pi}^{\pi} \cos kx \cos nx\,dx + b_k \int_{-\pi}^{\pi} \sin kx \cos nx\,dx \right)$$

By old HW we know all these integrals are zero except $\int_{-\pi}^{\pi} \cos^2 nx \, dx = \pi$. Thus

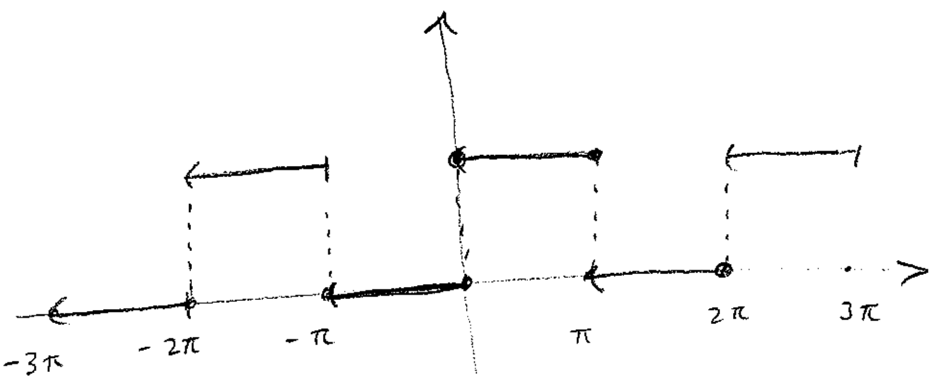
$$\int_{-\pi}^{\pi} f(x) \cos nx \, dx = a_n \pi$$

Thus and similarly

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

Ex: Square wave: $f(x) = \begin{cases} 0 & \text{if } -\pi < x \le 0 \\ 1 & \text{if } 0 < x \le \pi \end{cases}$

Periodic elsewhere



$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx = \frac{1}{\pi} \left(\int_{-\pi}^{0} 0 \, dx + \int_{0}^{\pi} 1 \, dx \right) = \underline{1}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 0 \cos nx\, dx + \int_{0}^{\pi} \cos nx\, dx \right)$$

$$= \frac{1}{\pi}\left(\frac{1}{n}\sin nx \Big|_{x=0}^{x=\pi} \right) = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\int_{0}^{\pi} \sin nx\, dx$$

$$= \frac{1}{\pi n}(-\cos nx)\Big|_{x=0}^{x=\pi} = \begin{cases} 0 & n \text{ even} \\ \frac{2}{\pi n} & n \text{ odd} \end{cases}$$

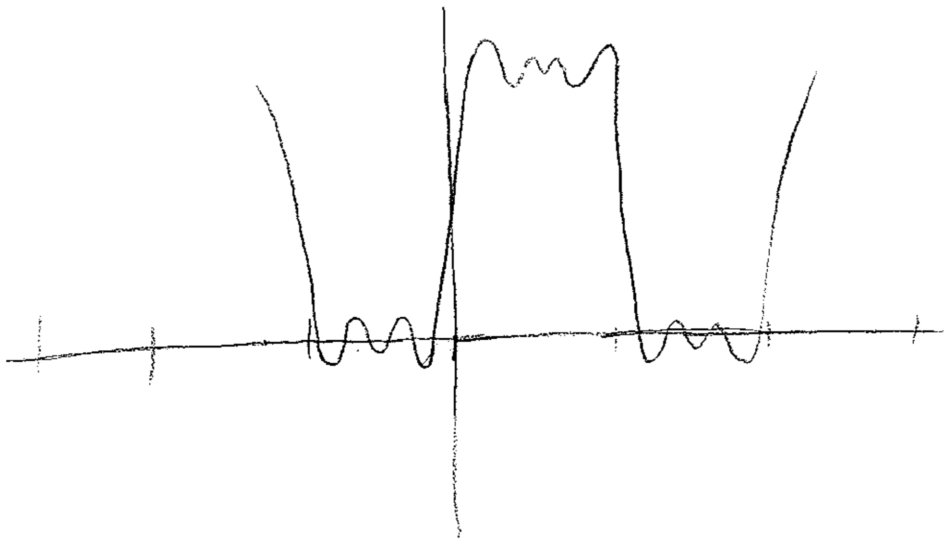
So the Fourier series is

$$\frac{1}{2} + \sum_{k=0}^{\infty} \frac{2}{(2k+1)\pi} \sin((2k+1)x)$$

Does this converge to our original f? E.g. consider

$$F_n(x) = \frac{1}{2} + \sum_{k=0}^{n} \frac{2}{(2k+1)\pi} \sin((2k+1)x)$$

F_3



In fact $F_n(x) \longrightarrow f(x)$ as $n \longrightarrow \infty$ except for $x = k\pi$ for some integer k, where it conv to $1/2$.

Fourier Convergence Theorem: f a fn, 2π-periodic

If f is continuous except for finitely many jumps, then its Fourier series converges to f except at the jumps.

Other periods: If f has period $T = 2L$ then use

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos \frac{kx}{L} + b_k \sin \frac{kx}{L} \right)$$

where

$$a_k = \frac{1}{L} \int_{-L}^{L} f(x) \cos kx \, dx$$

$$b_k = \frac{1}{L} \int_{-L}^{L} f(x) \sin kx \, dx .$$

Wavelets, etc: JPEG 2000.