Lecture 33: Properties of curves in polar coordinates (§9.5)

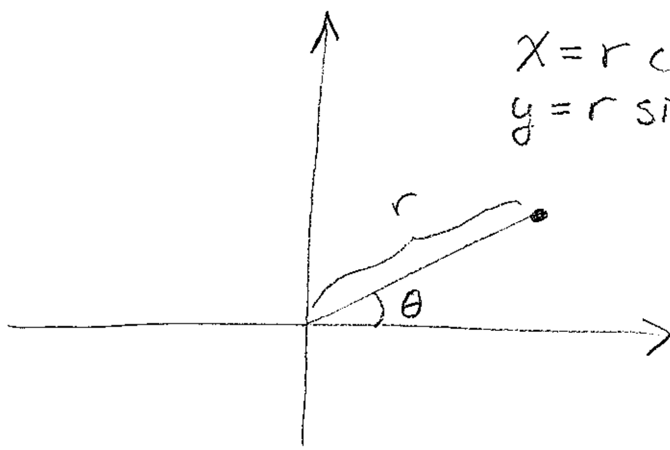
HW: §9.5 # 9, 13, 15, 20, 39

Next time: §9.6

Last time:

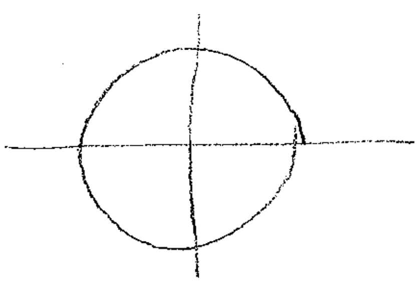
Polar coordinates:

$x = r\cos\theta$
$y = r\sin\theta$

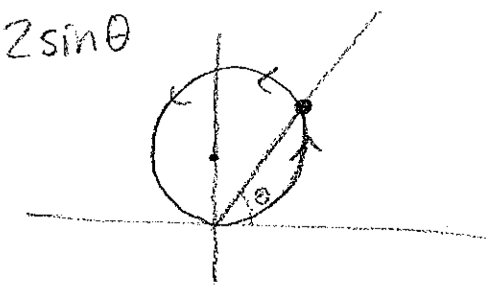


Plane curves:

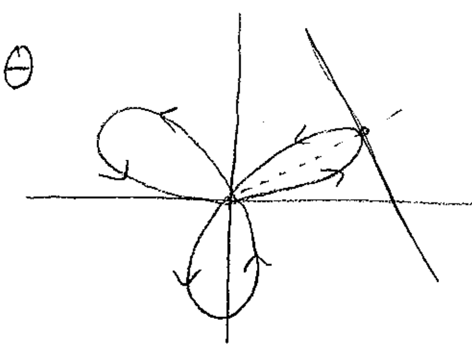
$r = 2$



$r = 2\sin\theta$



$r = \sin 3\theta$



Tangent lines:

Ex: Find the slope of the tangent line to the curve $r = \sin 3\theta$ at $\theta = \pi/6$

$$x(\theta) = r\cos\theta = \sin 3\theta \cos\theta$$

$$y(\theta) = r\sin\theta = \sin 3\theta \sin\theta$$

So

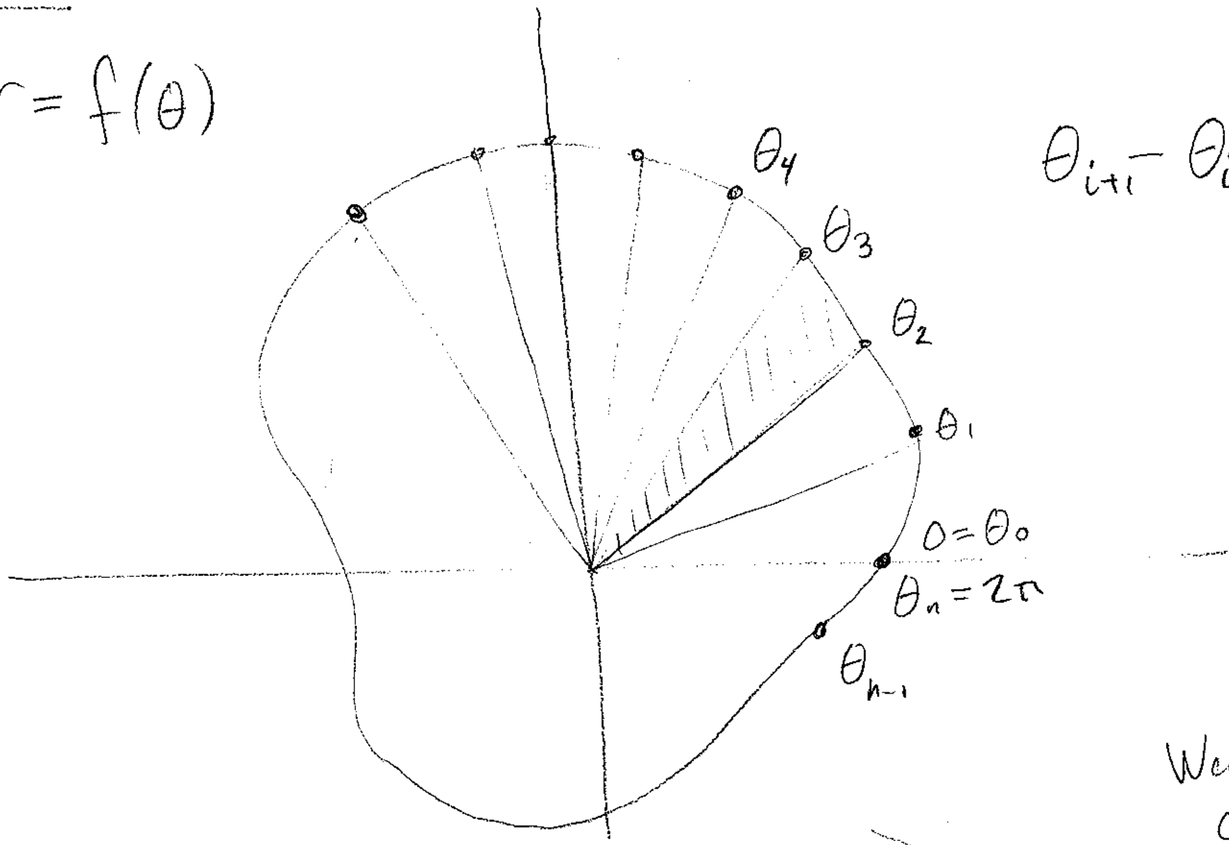
$$\text{slope} = \frac{y'(\pi/6)}{x'(\pi/6)} = \left.\frac{3\cos 3\theta \sin\theta + \sin 3\theta \cos\theta}{3\cos 3\theta \cos\theta - \sin 3\theta \sin\theta}\right|_{\theta=\pi/6}$$

$$= \frac{\cos \pi/6}{\sin \pi/6} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$$

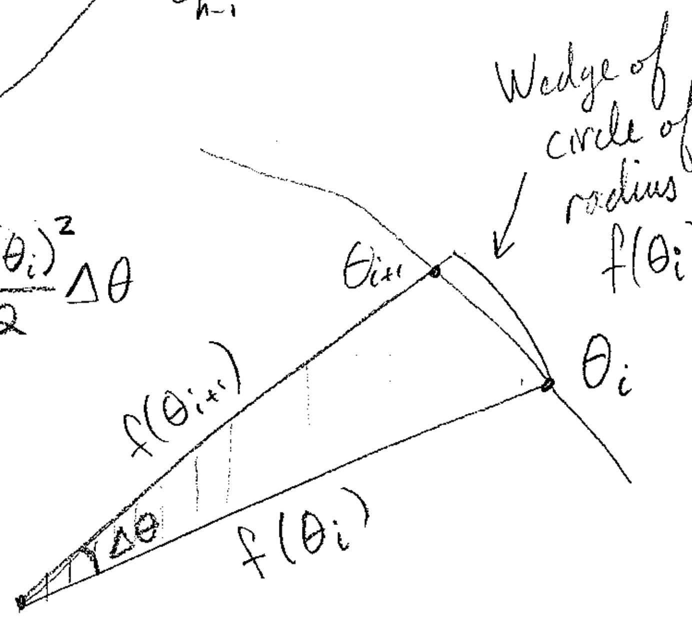
Area:

$r = f(\theta)$

$$\theta_{i+1} - \theta_i = \Delta\theta$$



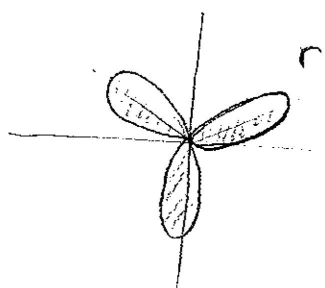
Area of ith wedge $\approx$ Area of circular wedge of rad $f(\theta_i)$ angle $\Delta\theta$ $= \dfrac{f(\theta_i)^2}{2}\Delta\theta$

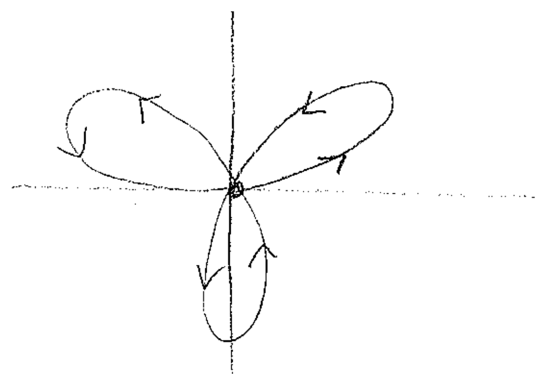


$$\text{Area} = \sum \text{Area of } i^{th} \text{ wedge} \approx \underbrace{\sum_{i=0}^{n-1} \frac{f(\theta_i)^2}{2} \Delta\theta}_{\text{Riemann Sum}} \approx \int_0^{2\pi} \frac{f(\theta)^2}{2} d\theta$$

So as $\Delta\theta \to 0$

$$\text{Area} = \int_0^{2\pi} \frac{f(\theta)^2}{2} d\theta$$

Ex:  Find area inside clover leaf.



Note: Need to be careful; doing the each loop only takes time $\pi/3$

$$\text{Area} = 3\begin{pmatrix}\text{Area}\\ \text{of}\\ \text{one leaf}\end{pmatrix} = 3\int_0^{\pi/3} \frac{1}{2}\sin^2 3\theta \, d\theta = 3\int_0^{\pi/3} \frac{1}{4}(1-\cos 6\theta)\, d\theta$$

$$= \frac{3}{4}\left(\theta - \frac{1}{6}\sin 6\theta\right)\Big|_{\theta=0}^{\pi/3} = \frac{\pi}{4}$$

Arc Length: $\int_a^b \sqrt{(x'(t))^2+(y'(t))^2}\,dt$

Suppose we have a polar curve given by $r=f(\theta)$. Then arclength formula becomes

$$\int_a^b \sqrt{(f'(\theta))^2+(f(\theta))^2}\,d\theta$$

since

$$x(\theta) = r\cos\theta = f(\theta)\cos(\theta)$$
$$y(\theta) = r\sin\theta = f(\theta)\sin(\theta)$$

and so $(x'(t))^2+(y'(t))^2 = (f'(\theta)\cos\theta - f(\theta)\sin\theta)^2 + (f'(\theta)\sin\theta + f(\theta)\cos\theta)^2$

$$= (f'(\theta))^2\cos^2\theta + (f(\theta))^2\sin^2\theta + f'(\theta))^2\sin\theta + f(\theta))^2\cos^2\theta$$

$$= (f'(\theta))^2 + (f(\theta))^2.$$