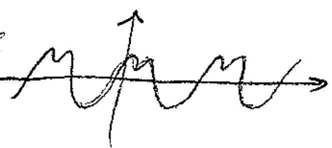
Lecture 31: Midterm Review

Exam Friday: bring 1 sheet of notes. / Review Prob. on Web.

Extra Office Hours: Today 3–5 pm; Thursday 9–11 am

Understand / Approximating functions: $f(x)$

$f(x)$ infinitely diff near $c \rightsquigarrow$ Power/Taylor series

$f(x)$ periodic  $\rightsquigarrow$ Fourier series

Given f, how do we find its Taylor series?

Directly: $$\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

Compute $f^{(k)}(c)$, find pattern

Using known series: $\frac{1}{1-x}$, e^x, $\sin x$, $\ln x$, ...

Subsition: $$e^{x^2} = \sum_{k=0}^{\infty} \frac{1}{k!}(x^2)^k = \sum_{k=0}^{\infty} \frac{1}{k!} x^{2k}$$

Rewriting: $\frac{1}{1+x} = \frac{1}{1-(-x)}$

Slight Changes: $$x e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^{k+1}$$

$\boxed{\text{Integration / Differentiation}}$

Ex: 8.6 #33 $\quad f(x) = \dfrac{2x}{(1-x^2)^2}$ Taylor series around $c=0$.

Notice

$$\frac{d}{dx}\Big(\underbrace{\frac{1}{1-x^2}}_{g(x)}\Big) = \frac{2x}{(1-x^2)^2} \quad \text{and } g(x) = \frac{1}{1-(x^2)} = \sum_{k=0}^{\infty} x^{2k}$$

$$= 1 + x^2 + x^4 + x^6 + \dots$$

So $f(x) = g'(x) = \underbrace{\sum_{k=1}^{\infty}}_{\to \text{ why?}} (2k)x^{2k-1} = 2x + 4x^3 + 6x^5 + \dots$

Directly: $f(0) = 0 \qquad f'(x) = \dfrac{2(1-x^2)^2 + 8x^2(1-x^2)}{(1-x^2)^4} =$

$f'(0) = 2 \qquad f''(x) = \dfrac{24x}{(1-x^2)^3} + \dfrac{48x^3}{(1-x^2)^4}$

$f''(0) = 0$

$f'''(0) = 24 \qquad f''' = \dfrac{24}{(1-x^2)^3} + \dfrac{288x^2}{(1-x^2)^4} + \dfrac{384x^4}{(1-x^2)^5}$

What a mess....

Note: Radius of convergence of series for f is the same as that of g; i.e. $\underline{1}$. (Diverges at $x = \pm 1$)

Taylor series converges to $f(x)$ on $(-1, 1)$ by construction

<u>Error estimates</u>: Suppose f has Taylor series $\sum_{k=0}^{\infty} b_k (x-c)^k$ which converges at x.

<u>Q1</u>: For this fixed x, is $\sum_{k=0}^{\infty} b_k (x-c)^k = f(x)$?

<u>Q2</u>: For this fixed x, how close is $P_n(x) = \sum_{k=0}^{n} b_k (x-c)^k$ to $f(x)$?

Interested in $R_n(x) = f(x) - P_n(x)$.

For the 1st question, yes $\iff R_n(x) \to 0$ as $n \to \infty$

2nd question, estimate $R_n(x)$ for fixed n.

<u>Taylor's Theorem</u>: f smooth enough. Then

$$R_n(x) = \underbrace{\frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}}$$ for some z in between c and x

basically, the next term in the Taylor except have $f^{(n+1)}(z)$ instead of $f^{(n+1)}(c)$

<u>Estimate</u> $\cos \frac{1}{2}$ to an accuracy of 10^{-5}

Taylor series of $\cos x$ about 0 is $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = 1 - \frac{x^2}{2} + \cdots$

$$P_n(x) = \sum_{k=0}^{n/2} \frac{(-1)^k}{(2k)!} x^{2k}$$

terms through x^n

$$P_2 = 1 - \frac{x^2}{2}$$

$$P_3 = 1 - \frac{x^2}{2}$$

$$P_4 = 1 - \frac{x^2}{2} + \frac{x^4}{4!}$$

Now

$\pm\cos x$ or $\pm\sin x$

$$\left| R_n(x) \right| = \left| \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1} \right| \leq \frac{|x|^{n+1}}{(n+1)!}$$

and $n = 6$ and $x = 1/2$ gives

$$\left| R_7(1/2) \right| \leq \frac{(1/2)^8}{8!} = 1.5 \times 10^{-6}$$

So

$$P_7(1/2) = \overbrace{1 - \frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{4!}\left(\frac{1}{2}\right)^4 - \frac{1}{6!}(1/2)^6}^{\star} = 0.87758246\ldots$$

is within 1.5×10^{-6} of $\cos 1/2$.

<u>Alternate Method</u>: By HW, we <u>know</u> $\cos\frac{1}{2} = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\left(\frac{1}{2}\right)^{2k}$

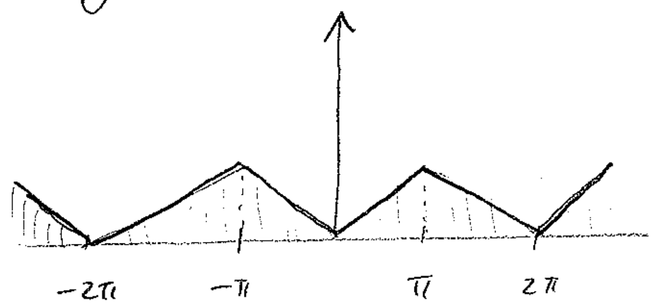
which is an <u>alternating series</u> so $\star$

differs from the total sum $\cos\frac{1}{2}$ by at most the

next term: $\frac{1}{8!}(1/2)^8 = 9.7 \times 10^{-8}$. [Better estimate, but relies on us knowing the ans. to Q1.]

Fourier Series: Consider $f(x)$ to be $|x|$ on $-\pi \le x \le \pi$, repeating periodically elsewhere

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos kx + b_k \sin kx \right)$$



As f is <u>even</u> all $b_k = 0$.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{1}{\pi} x^2 \Big|_0^{\pi}$$

since integrand is even

$$= \pi$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos kx\,dx = \frac{2}{\pi}\int_0^{\pi} x \cos kx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \frac{d}{dx}\frac{\sin kx}{k}\,dx = \frac{2}{\pi k}\int_0^{\pi} x \frac{d}{dx}\sin kx\,dx$$

$$= \frac{2}{\pi k}\left(x \sin kx \Big|_0^{\pi} - \int_0^{\pi} \sin kx\,dx \right) = \frac{2}{\pi k}\left(\frac{\cos kx}{k} \right)\Big|_0^{\pi}$$

$$= \frac{2}{\pi k^2}\Big(\cos k\pi - \underbrace{\cos 0}_{1} \Big) = \begin{cases} \dfrac{-4}{\pi k^2} & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$$

So, Fourier Series is $\frac{\pi}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}} \frac{-4}{\pi k^2} \cos kx$

$$= \frac{\pi}{2} + \sum_{n=0}^{\infty} \frac{-4}{\pi(2n+1)^2} \cos(2n+1)x$$

Q1: For which x does $\uparrow$ equal $f(x)$?

Ans: By Fourier Conv. Theorem, all of them as f is continuous.

Q2: What about error? We didn't study this...

Fun Formula: Take $x = \pi$

$$f(\pi) = \pi = \frac{\pi}{2} + \sum_{n=0}^{\infty} \frac{-4}{\pi(2n+1)^2} \overbrace{\cos(2n+1)\pi}^{-1}$$

$$\Rightarrow \frac{\pi^2}{8} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$$