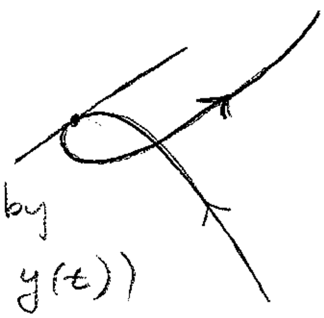
Lecture 30: Properties of Plane Curves

HW: (Nov 12): §9.2 #21, 26 §9.3 # 5, 7.

Midterm Friday, send me topics for Wed. Review
on §8.6–9.3 (through pg 737)

Extra Office Hours: Wed 3–5:00 Thur 9–11

Tangent lines:

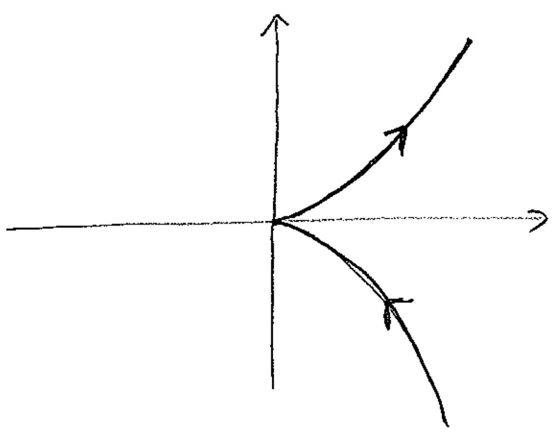


Param. by
$(x(t), y(t))$

$$\begin{array}{c}\text{Slope}\\ \text{of}\\ \text{tangent}\\ \text{line}\\ \text{at } t_0\end{array} = \frac{dy}{dx}(t_0) = \frac{y'(t_0)}{x'(t_0)}$$

If both $y'(t_0)$ and $x'(t_0)$ are 0, typically no tangent line.

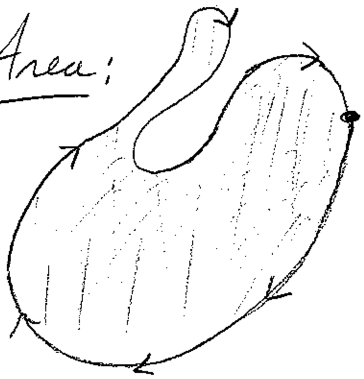
Ex: $x = t^2$, $y = t^3$ $\quad x'(0) = 2t\big|_{t=0} = 0 \quad y'(0) = 3t^2\big|_{t=0} = 0.$



Notes: This curve is also given by $x^3 - y^2 = 0$.

If $x \geq 0$ we have $t = \sqrt{x}$ and $y = x^{3/2}$, which explains the graph.

Area:



Suppose

$x(t),\ y(t)$ for $a \le t \le b$

trace out a closed curve once

(i.e. curve does not intersect itself except for $(x(a), y(a)) = (x(b), y(b))$)

Clockwise:

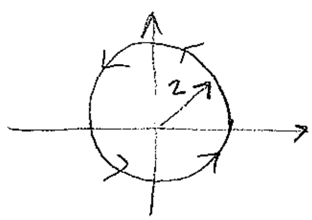
$$\text{Area} = \int_a^b y(t)\,x'(t)\,dt = -\int_a^b x(t)\,y'(t)\,dt$$

Counterclockwise:

$$\text{Area} = -\int_a^b y(t)\,x'(t)\,dt = \int_a^b x(t)\,y'(t)\,dt$$

[Why does this work? Green's Theorem from Vector Calc...
Fund. Thm's Calc in higher dims...]

Ex:



$x(t) = 2\cos t$

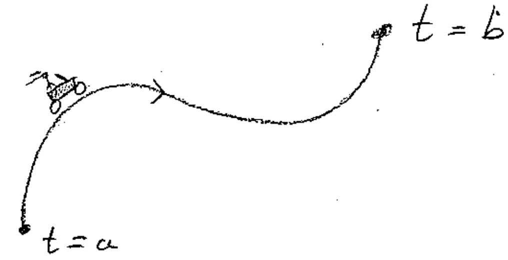
$y(t) = 2\sin t$ $\qquad y'(t) = 2\cos t$

$0 \le t \le 2\pi$

$$\text{Area} = \int_0^{2\pi} x(t)\,y'(t)\,dt = 4\int_0^{2\pi} \cos^2 t\,dt =$$

$$= 4\int_0^{2\pi} \tfrac{1}{2}(1+\cos 2t)\,dt = 4\pi = \pi(2)^2 \text{ as expected.}$$

Length of a curve:



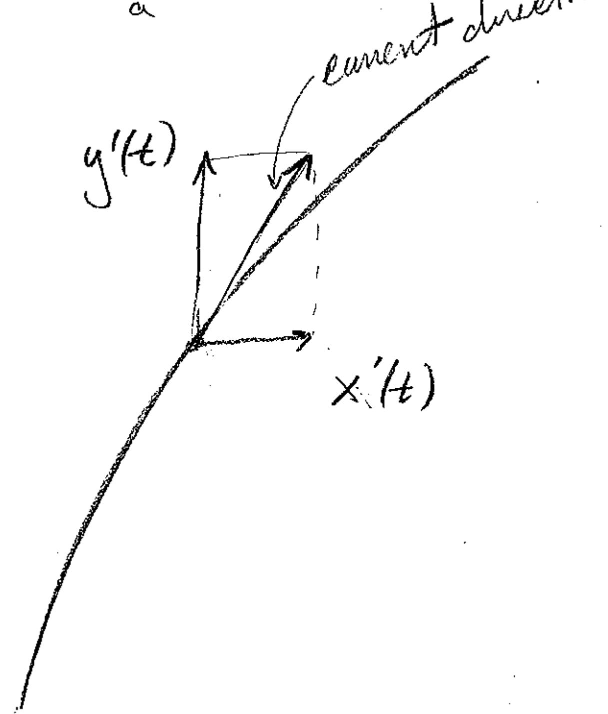
(dist) = (rate)(time) $\leadsto$

$V(t)$ = velocity at time t

$= \frac{d}{dt}(\text{dist})$

$$\begin{matrix}\text{total}\\ \text{dist}\\ \text{travelled}\end{matrix} = \int_a^b V(t)\,dt$$

Suppose curve is parametrized by $(x(t), y(t))$. What is $V(t)$?

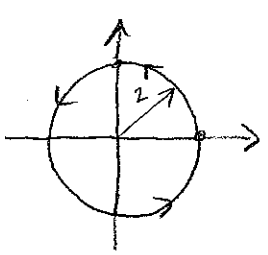


$$V(t) = \sqrt{(x'(t))^2 + (y'(t))^2}$$

So:

$$\begin{matrix}\text{Length of}\\ \text{curve}\end{matrix} = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

Ex:



$$V(t) = \sqrt{(-2\sin t)^2 + (2\cos t)^2} = 2$$

$x(t) = 2\cos t$
$y(t) = 2\sin t$ for $0 \le t \le 2\pi$

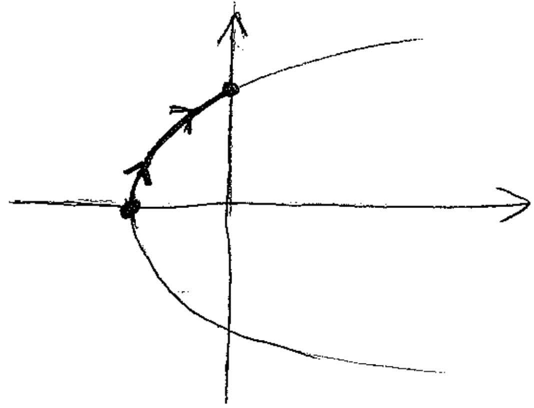
$$\text{Length} = \int_0^{2\pi} v(t)\,dt = \int_0^{2\pi} 2\,dt = 4\pi$$

$= 2\pi(2)$ as expected.

Ex: $x = t^2 - 1$

$y = t$

$0 \leq t \leq -1$



$$\text{Length} = \int_0^1 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = \int_0^1 \sqrt{(2t)^2 + 1}\, dt$$

Now: $\displaystyle\int \sqrt{(2t)^2+1}\, dt = \int \frac{1}{2}\sec^3\theta\, d\theta \overset{\text{reduction formula}}{=}$

$2t = \tan\theta$

$\sec\theta$

$dt = \frac{1}{2}\sec^2\theta\, d\theta$

$$\frac{1}{2}\left(\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\int \sec\theta\, d\theta\right)$$

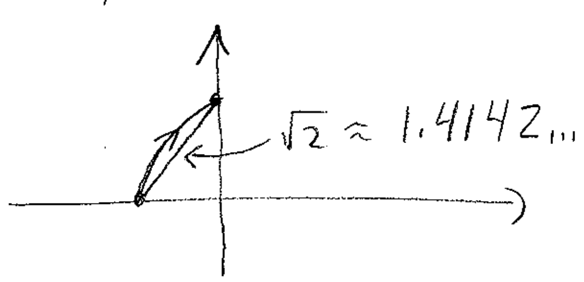
$$= \frac{1}{4}\sec\theta\tan\theta + \frac{1}{4}\ln|\sec\theta + \tan\theta| + C$$

$$= \frac{1}{2}t\sqrt{(2t)^2+1} + \frac{1}{4}\ln\left|\sqrt{(2t)^2+1} + 2t\right| + C$$

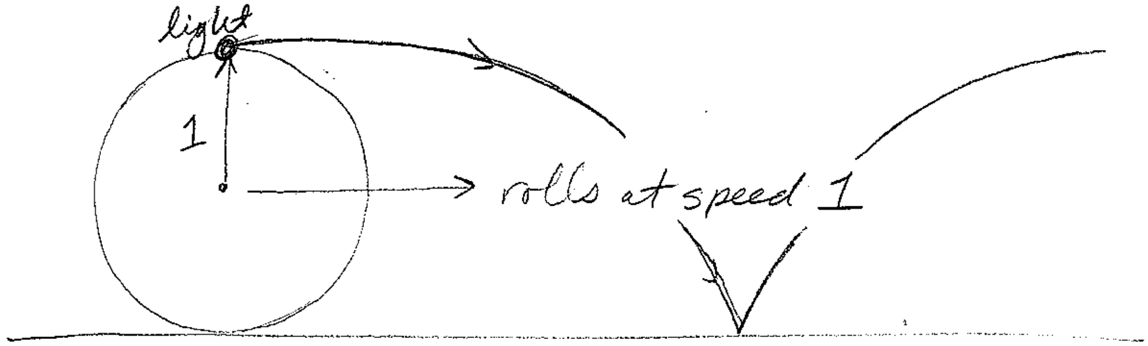
So; Length = $\Big|_0^1 = \frac{1}{2}\sqrt{5} + \frac{1}{4}\ln(\sqrt{5}+2) - 0$

$$= \frac{1}{2}\sqrt{5} + \frac{1}{4}\ln(\sqrt{5}+2) \approx 1.478942\ldots$$

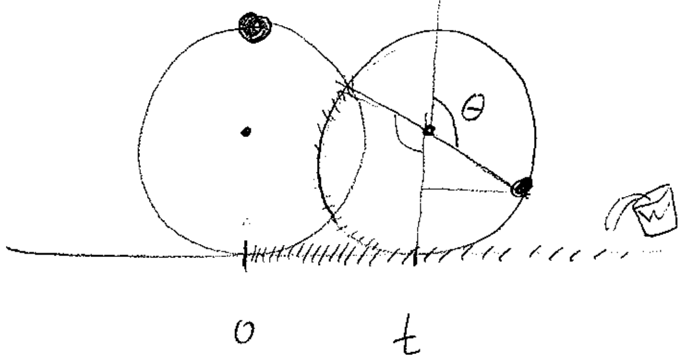
Compare to diagonal path



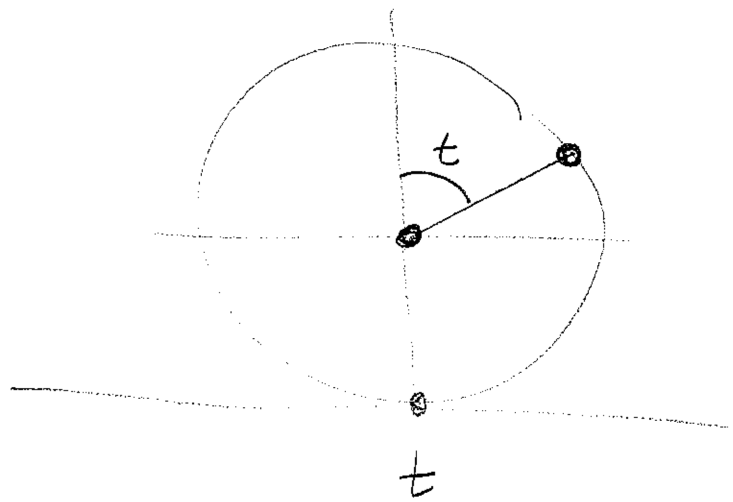
Cycloid:



After time t, how much has the light rotated?



Ans: $\theta = t$



So
$$x(t) = t + \sin t$$
$$y(t) = 1 + \cos t$$