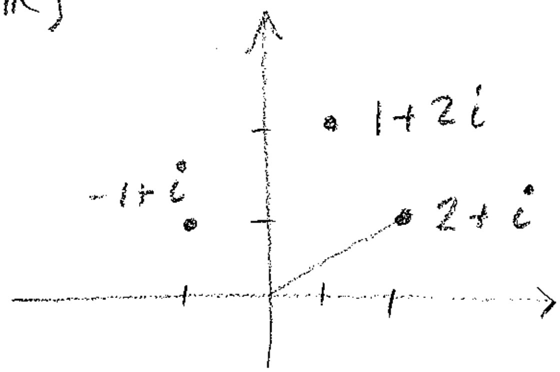
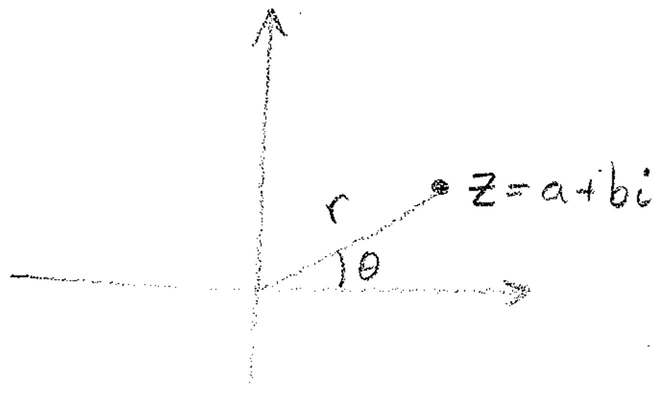
Lecture 38: Complex Numbers II

Last time: $\mathbb{C} = \{ a+bi \mid a, b \text{ in } \mathbb{R} \}$

$$|a+bi| = \sqrt{a^2+b^2}$$



Polar coordinates



$$\boxed{\text{Key: } z = r\underbrace{e^{i\theta}}}$$

But what does this mean?

Suppose θ is a real number. Then as $e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$

$$e^{i\theta} = \sum_{k=0}^{\infty} \frac{1}{k!} (i\theta)^k$$

	1	2	3	4	5	6	7	8	
i^k	i	-1	$-i$	1	i	-1	$-i$	1	...

$$= 1 + i\theta + \frac{1}{2!}(i\theta)^2 + \frac{1}{3!}(i\theta)^3 + \frac{1}{4!}(i\theta)^4 + \frac{1}{5!}(i\theta)^5 + \frac{1}{6!}(i\theta)^6 + \cdots$$

$$= 1 + \theta i - \frac{1}{2!}\theta^2 - \frac{1}{3!}\theta^3 i + \frac{1}{4!}\theta^4 + \frac{1}{5!}\theta^5 i - \frac{1}{6!}\theta^6 - \frac{1}{7!}\theta^7 i + \frac{1}{8!}\theta^8 + \cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right)$$

$= \cos\theta + i\sin\theta$. So $\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$

and $re^{i\theta} = r\cos\theta + i(r\sin\theta) = (\underset{x}{\underset{\shortparallel}{r\cos\theta}}, \underset{y}{\underset{\shortparallel}{r\sin\theta}})$

as claimed.

Consequences: $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$ $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

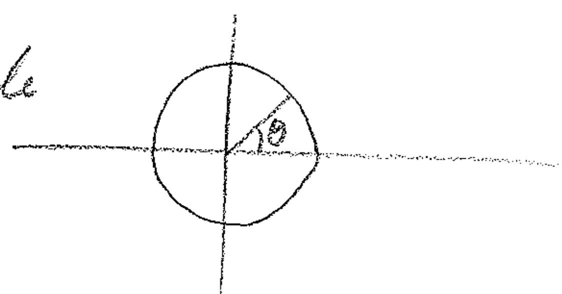
Fun with trig idents:

$$\underset{\cos 2\theta + i\sin 2\theta}{\underset{\shortparallel}{e^{i2\theta}}} = e^{i\theta}e^{i\theta} = (e^{i\theta})^2 = (\cos\theta + i\sin\theta)^2$$

$$= (\cos^2\theta - \sin^2\theta) + i(2\sin\theta\cos\theta)$$

Takeing real parts gives $\cos 2\theta = \cos^2\theta - \sin^2\theta$, etc

Note: $e^{i\theta}$ is on the unit circle



<u>Taking roots:</u> What are the cube roots of 1? that is, the solutions to $x^3 = 1$?

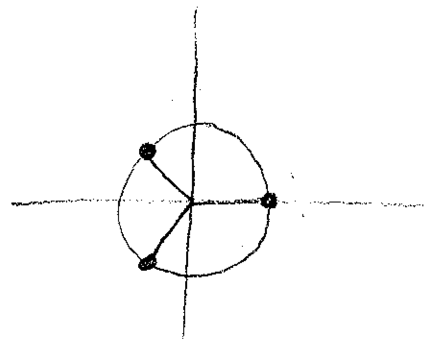
[Refer back to Cardano's Formula.]

Try $x = re^{i\theta}$, and then $1 = x^3 = (re^{i\theta})^3 = r^3 \underbrace{e^{i3\theta}}_{\text{has } |\,| = 1}$

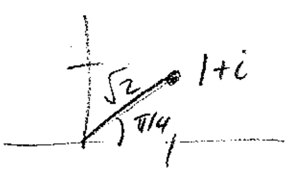
$\boxed{r, \theta \text{ are in } \mathbb{R}}$

Then $r^3 = 1 \Rightarrow r = 1$ and $3\theta = 2\pi k \Rightarrow \theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$

$\theta = 2\pi/3$, $x = \cos 2\pi/3 + i \sin \frac{2\pi}{3}$
$= -\frac{1}{2} + \frac{\sqrt{3}}{2} i$



<u>Ex</u>: Find $\sqrt{1+i}$. Now $1+i = \sqrt{2}\, e^{i\pi/4}$, and we



want $x = re^{i\theta}$ with $x^2 = r^2 e^{i2\theta} = 1+i$

So take $x = \sqrt[4]{2}\, e^{i\pi/8}$ or $x = \sqrt[4]{2}\, e^{i9\pi/8}$
$= -\sqrt[4]{2}\, e^{i\pi/8}$

In general, if $\alpha = re^{i\theta}$ then its n^{th} roots are

$$\sqrt[n]{r}\, e^{i(\theta/n + \frac{2\pi k}{n})} \quad \text{for } k = 0, \dots, n-1$$

Mandelbrot Set:

For c in $\mathbb{C}$, consider the sequence

$$c,\ c^2+c,\ (c^2+c)^2+c,\ \dots$$

that is $z_0 = c$ and $z_n = z_{n-1}^2 + c$

Ex: $c = 0$: $0, 0, 0, \dots \longrightarrow 0$

$c = 1$: $1, 2, 5, 26, 676, \dots \longrightarrow \infty$

$c = -\frac{1}{2} + \frac{1}{4}i$: $-\frac{1}{2} + \frac{1}{4}i,\ -\frac{5}{16},\ -\frac{103}{256} + \frac{i}{4},\ -\frac{26{,}255}{65\,536} + \frac{25}{512}i$

remains bounded

$$\mathcal{M} = \left\{ c \text{ in } \mathbb{C} \;\middle|\; \text{The sequence } |z_0|, |z_1|, |z_2|, |z_3|, \dots \text{ is bounded} \right\}$$

See picture on last page of Complex Handout.