Lecture 37: Complex Numbers

Handout: HW#12 + revised honors #4 + supplement on complex #5

Final: Friday, Dec 12 1:30–4:30 usual room.

[Complex numbers, as they relate to polar coor, series]

[Useful, indeed sometimes essential, tool, if initially a bit strange.
First studied by Cardano (16th cent) who called them "fictitious"]

Real Numbers: $\mathbb{R} = \{1, -2, 3/2, \pi, e, 1.0100100010000 1\ldots\}$

linear equations: $3x + 2 = 0 \quad \sqrt{2}x - \pi = 0$
exactly one solution

quadratic equation: $x^2 - 1 = 0$ has two solutions: $1, -1$

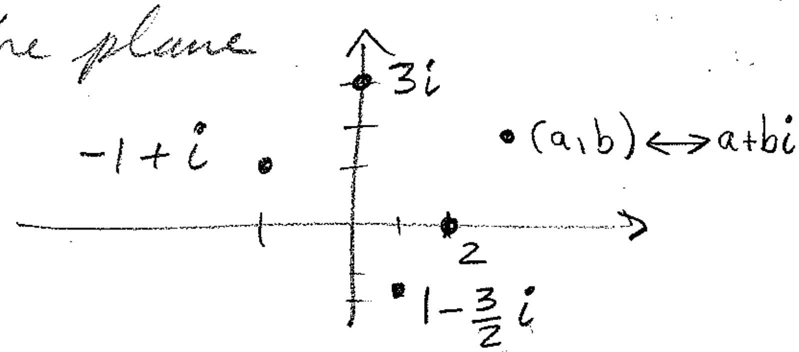
$x^2 + 1 = 0$ has no solutions in $\mathbb{R}$

Complex Numbers: add i where $i^2 = -1$ [solves]
i.e. $i = \sqrt{-1}$.

$\mathbb{C} = \{a + bi, \text{ where } a, b \text{ are in } \mathbb{R}\}$

Ex: $1 - \frac{3}{2}i$, 2, $3i$, $\pi - ei$, $\sqrt{2} - \sqrt{3}\,i$

Can identify $\mathbb{C}$ with the plane



Like $\mathbb{R}$ can add complex numbers:

$$(a+bi)+(c+di) = (a+b)+(b+d)i$$

Ex: $(1+i)+(-2+3i) = -1+4i$

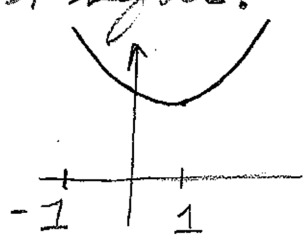
Ex: $(1+i)(-2+3i)$
$= (-2-3)+(3-2)i$
$= -5+i$

and multiply

$$(a+bi)(c+di) = (ac-bd)+(ad+bc)i$$

All the usual rules hold: commutative, associative, distributative...
[Also divison, as we'll see later.]

Allows us to solve equations we couldn't before:



Ex: x^2-2x+2 has no roots in $\mathbb{R}$,
but $x=1+i$ and $1-i$
are roots in $\mathbb{C}$: $(1+i)^2-2(1+i)+2$

$$= (1+2i-1)-2-2i+2 = 0.$$

Can find
using the quadratic formula: $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$$= \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm \sqrt{4}\sqrt{-1}}{2} = 1 \pm i$$

__Fundamental__
__Theorem of Algebra__

The equation

a_i in $\mathbb{C}$

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

has n roots in $\mathbb{C}$.

But why are complex numbers useful?

[Sure we can solve more equations, but what meaning do these new solutions have?]

<u>Original Motivation</u> (Italy 16th century)

Complex numbers as intermediate steps.

Due to del Ferro + Tartaglia

<u>Cardano's Formula</u>: $x^3 + cx + d = 0$ has solutions

$$x = t - \frac{c}{3t} \quad \text{where } t = \sqrt[3]{\frac{-d \pm \sqrt{d^2 + \frac{4}{27}c^3}}{2}}$$

<u>Ex</u>: $x^3 - 7x + 6 = 0$ roots: $-3, 1, 2$

$$t = \sqrt[3]{\frac{-6 \pm \sqrt{36 - \frac{4}{27}7^3}}{2}} = \sqrt[3]{\frac{-3 \pm \sqrt{-400/27}}{2}} = \sqrt[3]{-3 \pm \frac{10}{3\sqrt{3}}i}$$

For example

one of 3 possibilities

$$t = \sqrt[3]{-3 + \frac{10}{3\sqrt{3}}i} = 1 + \frac{2}{\sqrt{3}}i$$

and so a solution is

$$x = t + \frac{7}{3t} = 1 + \frac{2}{\sqrt{3}}i + \left(1 - \frac{2}{\sqrt{3}}i\right) = 2$$

Fact: There does not exist a cubic formula that does not pass through $\mathbb{C}$ for some cubics.

Simplification: trig formulas, Fourier series, transformations of the plane...

Physics: In Quantum Mechanics, particles are described by $\mathbb{C}$-valued "probability" distributions

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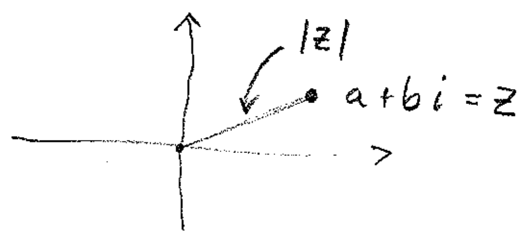
Terminology:

$$z = a + bi$$

a: real part, $\mathrm{Re}(z)$ b: imaginary part, $\mathrm{Im}(z)$

$|z|$ = absolute value of z $= \sqrt{a^2+b^2}$



$\overline{z}$ = complex conjugate $= a - bi$

$$z\overline{z} = |z|^2 \text{ since } (a+bi)(a-bi) = a^2+b^2+\overbrace{(-abi+abi)}^{0}$$

z 1/z

$$\frac{1}{z} = \frac{\overline{z}}{|z|^2} \quad \text{as} \quad (a+bi)\cdot\left(\frac{a-bi}{a^2+b^2}\right) = \frac{1}{a^2+b^2}(a^2+b^2) = 1.$$