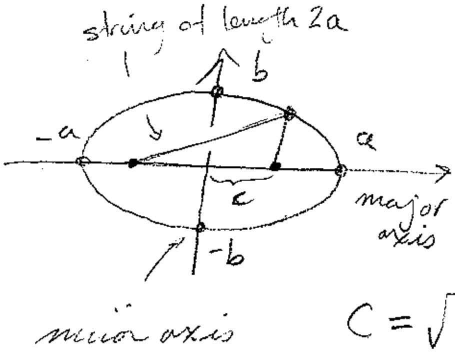
Lecture 35: Conic sections II

HW (Dec 3) §9.6: # 3, 7, 9, 32, 35, 42.

Next time: Complex Numbers

Last time:

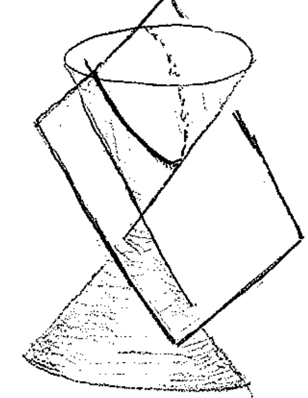
Conic sections: $ax^2 + bxy + cy^2 + dx + ey + f = 0$



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

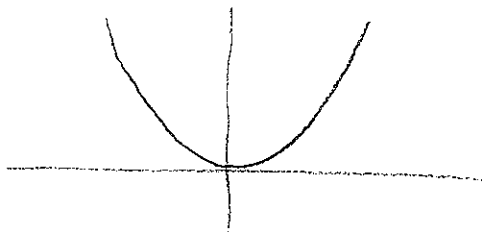
$\boxed{a > b \text{ shown}}$

$c = \sqrt{a^2 - b^2}$

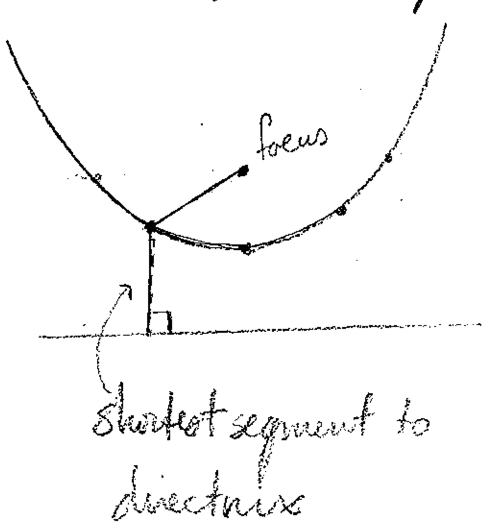


[More examples in text.]

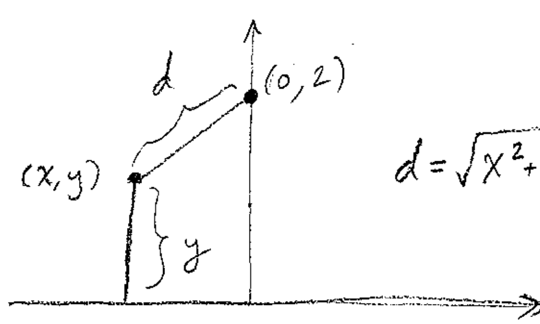
Parabola: Ex: $y = x^2$



Geom def: All points equidistant from the focus and directrix.



Ex: focus $= (0,2)$
directrix $=$ x-axis

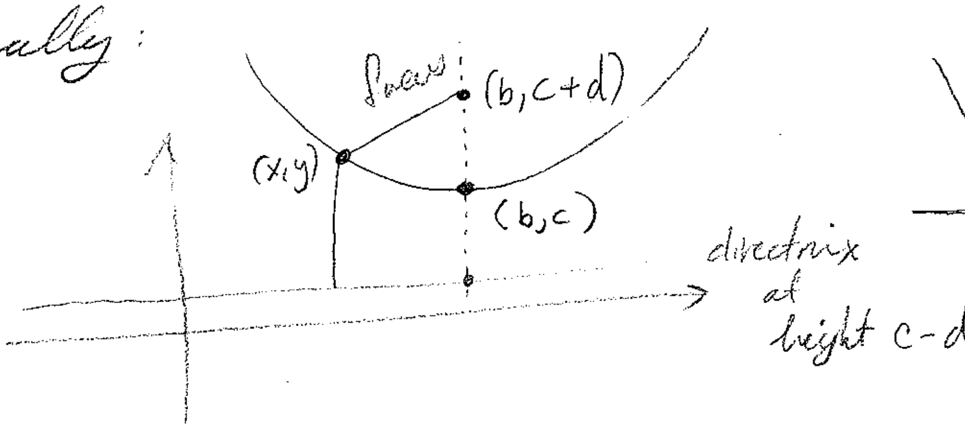


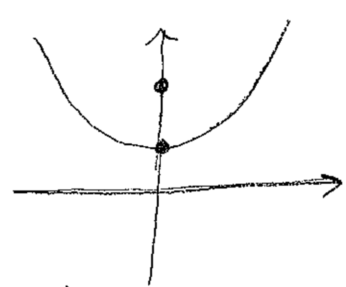
$d = \sqrt{x^2 + (y-2)^2}$

So the equation is $y = \sqrt{x^2+(y-2)^2}$ which is the same as

$$y^2 = x^2+(y-2)^2 \iff y^2 = x^2+y^2-4y+4 \iff 4y = x^2+4$$
$$\iff y = \tfrac{1}{4}x^2+1$$

More generally:





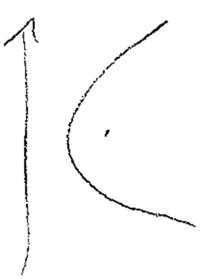
$$\text{Eqn} = \quad y-(c-d) = \sqrt{(x-b)^2+(y-(c+d))^2}$$

$$\iff y^2-2(c-d)y+(c-d)^2 = (x-b)^2+(y-(c+d))^2$$

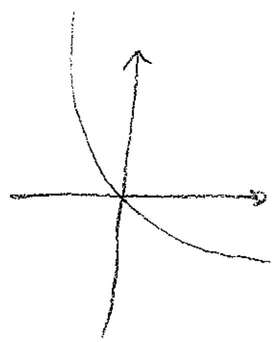
$$\iff y^2-2cy+2dy+c^2-2cd+d^2 = (x-b)^2+y^2-2yc-2dy+c^2+2cd+d^2$$

$$4dy = (x-b)^2+4cd \iff \boxed{y = \frac{1}{4d}(x-b)^2+c.}$$

Also have

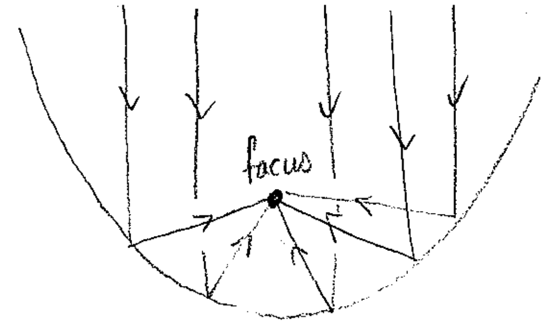


or



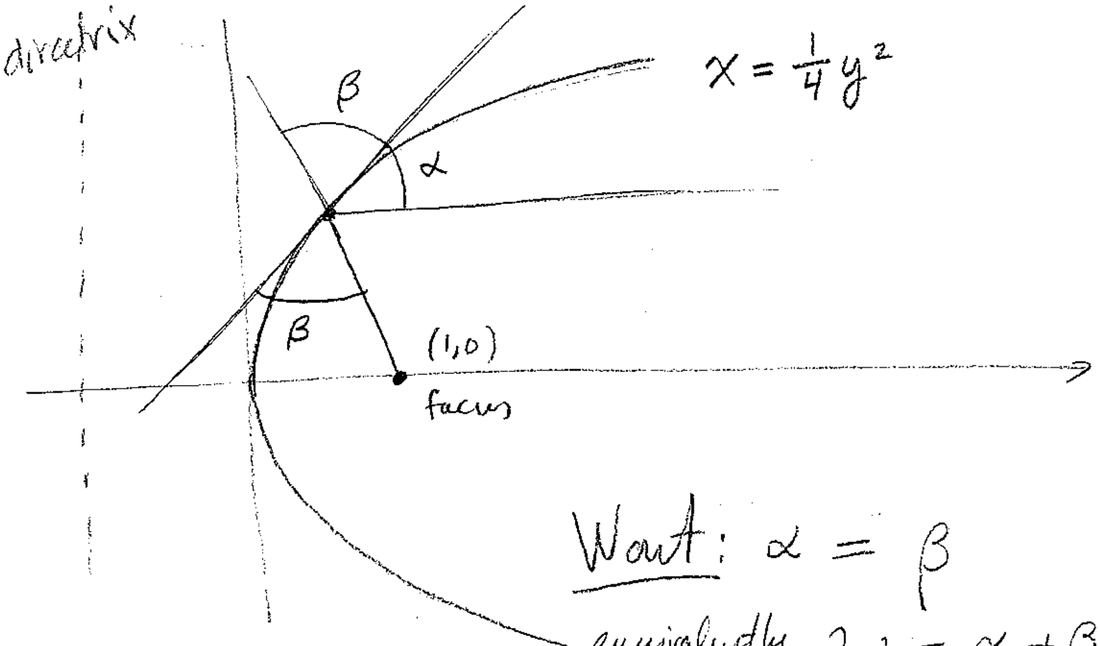
see text for details

Key Property:



Uses: radio telescope
solar thermal power
flashlight.

Reason:



$x = \frac{1}{4}y^2$

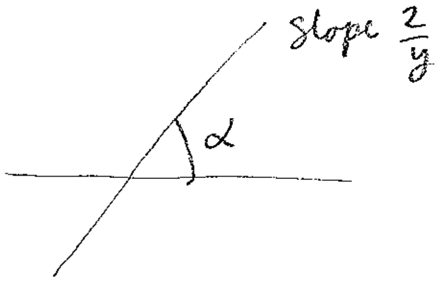
Want: $\alpha = \beta$

equivalently $2\alpha = \alpha + \beta$

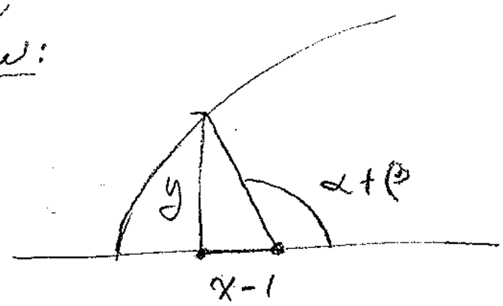
Slope of tangent line:

$$x = \frac{1}{4}t^2 \quad y = t \qquad \text{slope} = \frac{y'}{x'} = \frac{1}{\frac{1}{2}t} = \frac{2}{t} = \frac{2}{y}$$

So $\tan\alpha = \frac{2}{y}$



Now:



So $\tan(\alpha+\beta) = \frac{y}{x-1}$

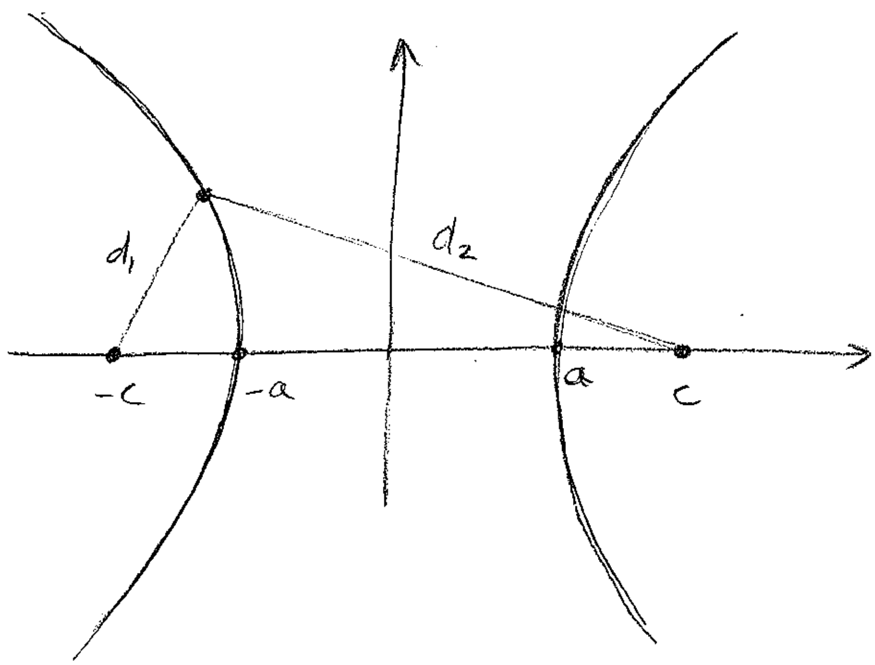
Point: $\tan 2\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha} = \dfrac{\frac{4}{y}}{1-\frac{4}{y^2}} \cdot \left(\dfrac{\frac{y^2}{4}}{\frac{y^2}{4}}\right) = \dfrac{y}{\frac{y^2}{4}-1}$

$$= \frac{y}{x-1} = \tan(\alpha+\beta).$$

$\boxed{\text{So } \alpha = \beta.}$

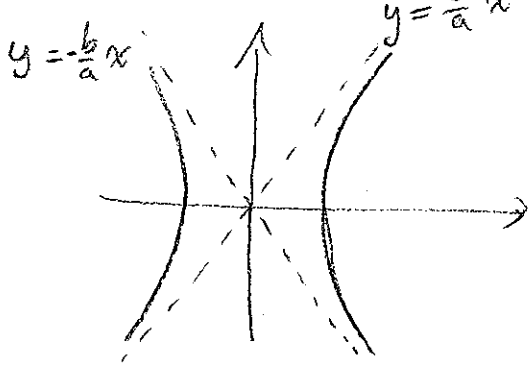
Hyperbola:

$|d_1 - d_2| = 2a$ ← fixed



Leads to $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ where $b^2 = c^2 - a^2$.

Ends of a hyperbola limit on two lines, called asymptotes. Reason:



$$\lim_{x\to\infty} \frac{y^2}{x^2} = \lim_{x\to\infty} \frac{\frac{b^2}{a^2}(x^2-1)}{x^2} = \frac{b^2}{a^2}$$

i.e. $\dfrac{y^2}{x^2} \approx \dfrac{b^2}{a^2} \Longleftrightarrow \dfrac{y}{x} \approx \pm\dfrac{b}{a}.$