Lecture 3: Integration by Parts (Section 6.2)

Turn in HW #1 • Monday is Labor Day

Office hours: Tue 2:30 – 4:00

HW #2: Due Wed Sept 3: Section 6.1 41, 44, 45
Section 6.2 7, 10, 19, 33, 37, 38.

Next time: Trig substitution (Section 6.3)

Section 6.1: Review of basic methods of integration, including substitution.

$$\int \sin x \cos x \, dx = \int u \, du = \frac{1}{2}u^2 = \frac{1}{2}\sin^2 x$$

$u = \sin x$
$du = \cos x \, dx$

Check: $\frac{d}{dx}\left(\frac{1}{2}\sin^2 x\right) = \sin x \cos x$

Integration rules come from differentiation rules:

$$\int f+g \, dx = \underbrace{\int f dx + \int g dx}$$

Means that $\longrightarrow$ is an anti-derivative for $f+g$:

$$\frac{d}{dx}\left(\int f dx + \int g dx\right) \overset{\text{sum rule}}{=} \frac{d}{dx}\left(\int f dx\right) + \frac{d}{dx}\left(\int g dx\right) = f + g$$

Where does subsitution come from? The chain rule:

$$\frac{d}{dx} g(u(x)) = \frac{dg}{dx}(u(x)) \frac{du}{dx}(x) = g'(u(x))u'(x)$$

[Thus if $\int f(x)\,dx = \int h(u(x)) \frac{du}{dx}(x)\,dx$ we can take $g = \int h(u)\,du$ and see $g(u(x))$ is an antiderivative of $f(x)$.]

Product rule: $$\frac{d}{dx}(u(x)v(x)) = \frac{du}{dx}(x)\,v(x) + u(x)\frac{dv}{dx}(x)$$

$$(uv)' = u'v + uv'$$

Integrating gives

$$uv = \int \frac{du}{dx} v\,dx + \int u \frac{dv}{dx}\,dx$$

and now rearrange to get "

Integration by parts

$$\int u \frac{dv}{dx}\,dx = uv - \int \frac{du}{dx} v\,dx$$

Ex: $$\int \underbrace{x}_{u} \underbrace{\cos x}_{\frac{dv}{dx}}\,dx = x\sin x - \int \overbrace{1}^{=\frac{du}{dx}} \cdot \sin x\,dx$$

where $v = \sin x$

$$= x\sin x + \cos x + C$$

Check: $$\frac{d}{dx}(x\sin x + \cos x) = \sin x + x\cos x - \sin x$$

$$= x\cos x$$

Why this worked: x gets simpler when we diff it, but $\sin x$ doesn't get more complicated when integrated

Compare:

$$\int \underset{\overset{\parallel}{\frac{dv}{dx}}}{x} \underbrace{\cos x}_{u} \, dx = (\cos x)\tfrac{1}{2}x^2 - \int \left(\tfrac{1}{2}x^2\right)(-\sin x)\,dx$$

$\frac{dv}{dx}$, that is, $v = \frac{1}{2}x^2$

$$= \tfrac{1}{2}x^2\cos x + \underbrace{\int \tfrac{1}{2}x^2 \sin x \, dx}_{\text{Blung...}}$$

Ex:

$$\int x^2 \sin x \, dx = \int x^2 \frac{d}{dx}(-\cos x)\,dx$$

$$= -x^2\cos x - \int \left(\frac{d}{dx}x^2\right)(-\cos x)\,dx$$

$$= -x^2\cos x + 2\int x\cos x\,dx$$

$$= -x^2\cos x + 2x\sin x + 2\cos x + C$$

Note: The book rewrites the key formula as

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

and then summarizes it as

$$\int u\,dv = uv - \int v\,du$$

Variations:

(a)
$$\int \ln x\,dx = \int 1 \cdot \ln x\,dx = \int \left(\frac{d}{dx}x\right)\ln x\,dx$$
$$= x\ln x - \int x\frac{d}{dx}(\ln x)\,dx = x\ln x - x + C$$

(b)
$$\int e^x\cos x\,dx = \int\left(\frac{d}{dx}e^x\right)\cos x\,dx$$
$$= e^x\cos x - \int e^x\left(\frac{d}{dx}\cos x\right)dx = e^x\cos x + \int e^x\sin x\,dx$$
$$= e^x\cos x + \int\left(\frac{d}{dx}e^x\right)\sin x\,dx = e^x\cos x + e^x\sin x - \int e^x\cos x\,dx$$

Thus
$$2\int e^x\cos x = e^x(\cos x + \sin x) \Rightarrow \int e^x\cos x = \frac{1}{2}e^x(\sin x + \cos x)$$

(c) Reduction formulae
$$\int x^n e^x\,dx = x^n e^x - n\int x^{n-1}e^x\,dx$$

(d)
$$\int_a^b u\frac{dv}{dx}\,dx = uv\Big|_{x=a}^{b} - \int_a^b \frac{du}{dx}\,v\,dx$$