

Lecture 3: Integration by Parts (Section 6.2)

(7)

Turn in HW #1 • Monday is Labor Day

Office hours: Tue 2:30 - 4:00

HW #2: Due Wed Sept 3^o Section 6.1 41, 44, 45
Section 6.2 7, 10, 19, 33,

Next time: Trig substitution (Section 6.3) 37, 38.

Section 6.1: Review of basic methods of integration, including substitution.

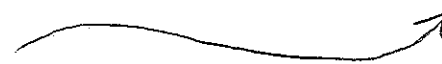
$$\int \sin x \cos x \, dx = \int u \, du = \frac{1}{2} u^2 = \frac{1}{2} \sin^2 x$$

$$u = \sin x \\ du = \cos x \, dx$$

Check: $\frac{d}{dx} \left(\frac{1}{2} \sin^2 x \right) = \sin x \cos x$

Integration rules come from differentiation rules:

$$\int f + g \, dx = \underbrace{\int f \, dx + \int g \, dx}$$

Means that  is an anti-derivative for $f + g$:

$$\frac{d}{dx} \left(\int f \, dx + \int g \, dx \right) \stackrel{\text{sum rule}}{=} \frac{d}{dx} \left(\int f \, dx \right) + \frac{d}{dx} \left(\int g \, dx \right) = f + g$$

Where does substitution come from? The chain rule:

$$\frac{d}{dx} g(u(x)) = \frac{dg}{du}(u(x)) \frac{du}{dx}(x) = g'(u(x)) u'(x)$$

[Thus if $\int f(x) dx = \int h(u(x)) \frac{du}{dx}(x) dx$ we can take $g = \int h(u) du$ and see $g(u(x))$ is an antiderivative of $f(x)$.]

Product rule: $\frac{d}{dx} (u(x)v(x)) = \frac{du}{dx}(x)v(x) + u(x)\frac{dv}{dx}(x)$

Integrating gives: $(uv)' = u'v + uv'$

$$uv = \int \frac{du}{dx} v dx + \int u \frac{dv}{dx} dx$$

and now rearrange to get "

Integration by parts

$$\int u \frac{dv}{dx} dx = uv - \int \frac{du}{dx} v dx$$

Ex: $\int \underbrace{x}_u \underbrace{\cos x}_{\frac{dv}{dx}} dx = x \sin x - \int 1 \cdot \overset{= \frac{du}{dx}}{\sin x} dx$
where $v = \sin x$ $= x \sin x + \cos x + C$

Check: $\frac{d}{dx} (x \sin x + \cos x) = \sin x + x \cos x - \sin x = x \cos x$

Why this worked: x gets simpler when we diff it, (8)
but $\sin x$ doesn't get more complicated
when integrated

Compare:

$$\int \underbrace{x}_u \underbrace{\cos x}_v dx = (\cos x) \frac{1}{2} x^2 - \int \left(\frac{1}{2} x^2 \right) (-\sin x) dx$$

$\frac{dv}{dx}$, that is, $v = \frac{1}{2} x^2$ $= \frac{1}{2} x^2 \cos x + \int \frac{1}{2} x^2 \sin x dx$

Blung...

Ex:

$$\begin{aligned} \int x^2 \sin x dx &= \int x^2 \frac{d}{dx} (-\cos x) dx \\ &= -x^2 \cos x - \int \left(\frac{d}{dx} x^2 \right) (-\cos x) dx \\ &= -x^2 \cos x + 2 \int x \cos x dx \\ &= -x^2 \cos x + 2x \sin x + 2 \cos x + C \end{aligned}$$

Note: The book rewrites the key formula as

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

and then summarizes it as

$$\int u dv = uv - \int v du$$

Variations:

$$\begin{aligned}\textcircled{a} \int \ln x \, dx &= \int 1 \cdot \ln x \, dx = \int \left(\frac{d}{dx} x \right) \ln x \, dx \\ &= x \ln x - \int x \frac{d}{dx} (\ln x) \, dx = x \ln x - x + C\end{aligned}$$

$$\begin{aligned}\textcircled{b} \int e^x \cos x \, dx &= \int \left(\frac{d}{dx} e^x \right) \cos x \, dx \\ &= e^x \cos x - \int e^x \left(\frac{d}{dx} \cos x \right) \, dx = e^x \cos x + \int e^x \sin x \, dx \\ &= e^x \cos x + \int \left(\frac{d}{dx} e^x \right) \sin x \, dx = e^x \cos x + e^x \sin x - \int e^x \cos x \, dx\end{aligned}$$

Thus

$$2 \int e^x \cos x = e^x (\cos x + \sin x) \Rightarrow \int e^x \cos x = \frac{1}{2} e^x (\sin x + \cos x)$$

Reduction formulae

$$\int x^n e^x \, dx = x^n e^x - n \int x^{n-1} e^x \, dx$$

$$\textcircled{d} \int_a^b u \frac{dv}{dx} \, dx = uv \Big|_{x=a}^b - \int_a^b \frac{du}{dx} v \, dx$$