Lecture 2: Foundations of Integration; Fund Thm 'o Calc

HW#1 (Due Friday) See syllabus

Class Notes on Web

Next time: Section 6.2

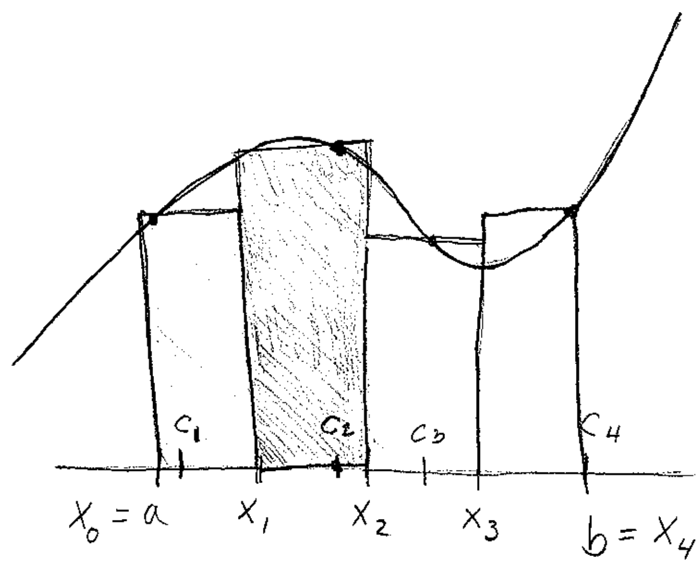
Last time:

Riemann Sum (= area of boxes)

$$\sum_{i=1}^{n} f(c_i)\,\Delta x$$

where $\Delta x = \frac{b-a}{n}$ and

c_i is in $[x_{i-1}, x_i]$



Def: A function f on $[a, b]$ has integral A if

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x = A$$ for every choice of sample points c_i

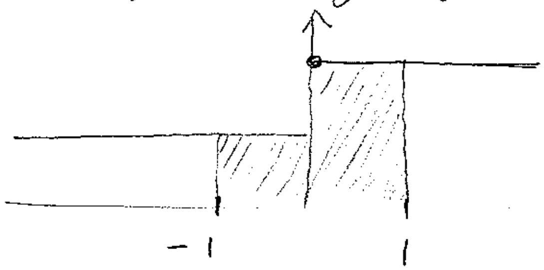
A nonintegrable function

$$f(x) = \begin{cases} 1 & \text{rational} \\ 0 & \text{irrational} \end{cases}$$

Thm: If f is continuous on $[a,b]$ then it is integrable.

Some discontinuous functions are integrable anyway

$$f(x) = \begin{cases} 2 & \text{if } 0 \le x \\ 1 & \text{otherwise} \end{cases}$$



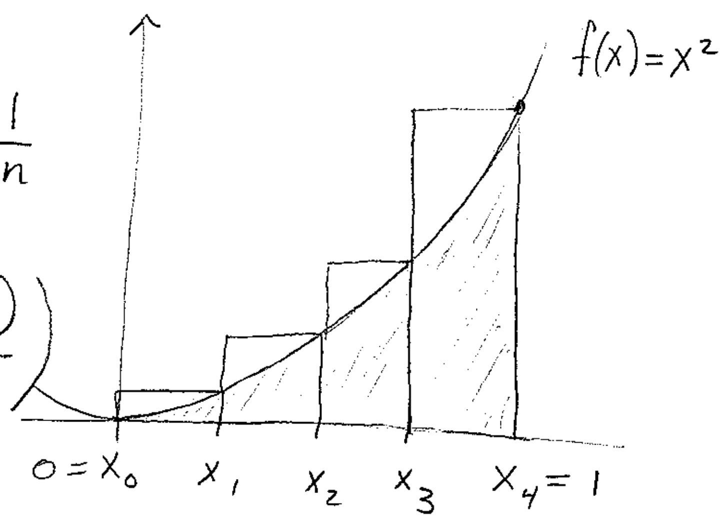
$$\int_{-1}^{1} f(x)\,dx = 3$$

[Technical note: We're using the Riemann integral here, but the Lebesgue integral can integrate just about anything. Mention applications: EE, fin. markets.]

Q: Who invented integration? A. Archimedes (286-211 BCE)

$$\text{Area} \approx \sum_{i=1}^{n} f(c_i)\Delta x = \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \frac{1}{n}$$

$$\frac{1}{n^3}\sum_{i=1}^{n} i^2 = \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)$$



[Compare $1+2+3+4+\cdots+n = \frac{n(n+1)}{2}$]

$$x_i = \frac{i}{n} \qquad \Delta x = \frac{1}{n}$$

$$c_i = x_i$$

As $n \to \infty$

$$\frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) = \frac{(1+1/n)(2+1/n)}{6} \longrightarrow 1/3$$

Thus $\int_0^1 x^2\,dx = 1/3$.

Of course, as modern persons, we usually use

<u>Fundamental Thm's Calculus</u>: f a continuous function on $[a,b]$.
If F is an antiderivative of f (i.e. $F'(x) = f(x)$) then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

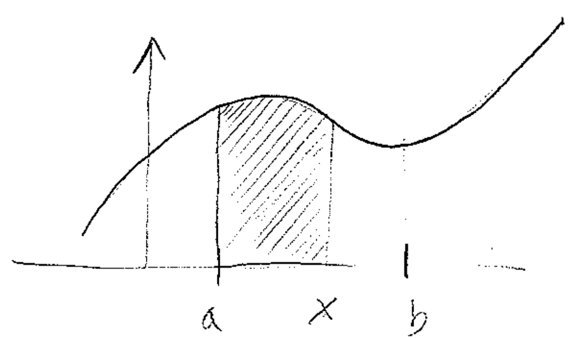
<u>E.g.</u>

$$\int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_{x=0}^{1} = \frac{1}{3} - 0 = 1/3$$

<u>Comments</u>: 1) So Newton-Leibnitz's key contribution was to realize integration was inverse of something namely differentiation.

2) Does every continuous function f on $[a,b]$ have an antiderivative F? Yes, namely you can take $F(x) = \int_a^x f(t)\,dt$

But its best when we have a nice formula for $F(x)$...



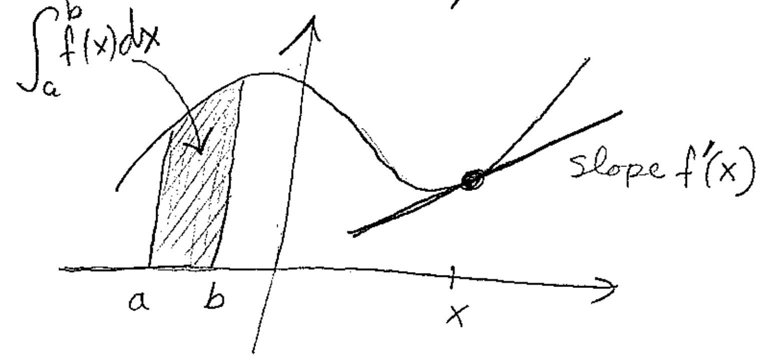
[Note to self: In fact, this + $\boxed{f' = 0 \Rightarrow f \text{ constant}}$ implies the Fund Thm 'o Calc]
consequence of MVT.

Heuristic for 2), if time permits

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\, dt$$

$$= \lim_{h \to 0} \begin{pmatrix} \text{Average of } f \\ \text{on } [x, x+h] \end{pmatrix} = f(x)$$

3) Geometric interpretations are not the only, or even primary, uses of calculus.

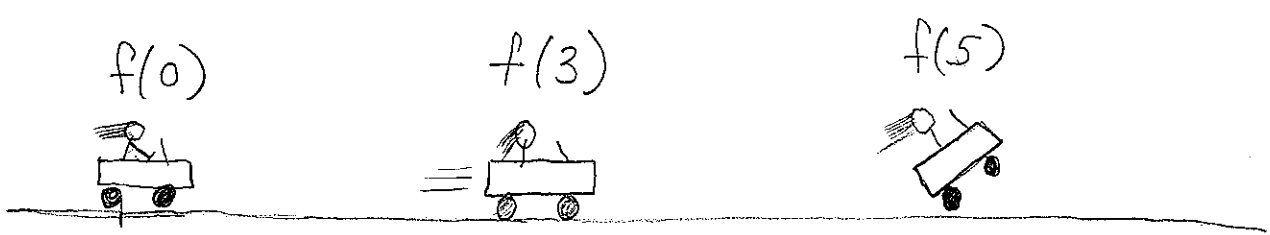


derivative = rate of change

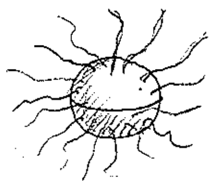
integral = accumulation of changes

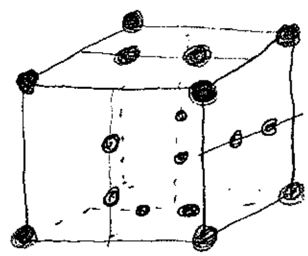
$f(t)$ = position at time t

$f'(t)$ = speed at time t



[Not the area/volume/etc as it sometimes important. Cf Randy Kamien's talk on crystalline lattices that are self-assembled by certain synthetic polymer molecules.





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Finding antiderivatives: What we know so far (§6.1)

Integration rule	Diff rule
$\int (f+g)\,dx = \int f\,dx + \int g\,dx$	$(f+g)' = f' + g'$
$\int f(u(x))\,u'(x)\,dx = \int f(u)\,du$	$\frac{d}{dx} f(g(x)) = \frac{df}{dx}(g(x))\,\frac{dg}{dx}(x)$ $= f'(g(x))\,g'(x)$
Integration by parts.	$(fg)' = f'g + f \cdot g'$