Math 231 E1H: Honors Problem Set 1

Due date: In class on Wednesday, September 24.

For each n, consider the quantity:

$$A_n = 2 \cdot \frac{2^2 \cdot 4^2 \cdot 6^2 \cdots (2n-2)^2 (2n)^2}{3^2 \cdot 5^2 \cdot 7^2 \cdots (2n-1)^2 (2n+1)}$$

or equivalently

$$A_n = 2 \cdot \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots \frac{2n}{2n-1} \cdot \frac{2n}{2n+1}$$

For instance, $A_1 = 8/3$ and $A_2 = 128/45$. In this honors set, you will show that $\lim_{n\to\infty} A_n = \pi$; this can be interpreted as saying that π is an infinite product

$$\pi = 2 \cdot \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots \frac{2n}{2n-1} \cdot \frac{2n}{2n+1} \cdots$$

1. Compute a decimal expression for A_1, A_2, A_3, and A_4. Do the numbers you get seem to be approach π? If you want, use a computer to find n where A_n agrees with π to several decimal places.

2. Consider the quantity $I_n = \int_0^{\pi/2} \sin^n(x)\, dx$. Compute I_0, I_1, and I_2.

3. Show that the relation $I_n = \frac{n-1}{n} I_{n-2}$ holds for $n \geq 2$. Hint: Use a problem from HW #2.

4. Show that $I_{2n+1}/I_{2n} = A_n/\pi$ for each $n \geq 1$.

5. Show that $\lim_{n\to\infty} I_{2n+1}/I_{2n} = 1$. Hint: First show that $I_{2n+2} \leq I_{2n+1} \leq I_{2n}$ by comparing the integrands. Then use the Squeeze Theorem on page 84.

6. Conclude that $\lim_{n\to\infty} A_n = \pi$.

Note: I'm certainly not asking for formal proofs but you should both show your work and explain your answer. Use of words is required, and complete sentences are encouraged.