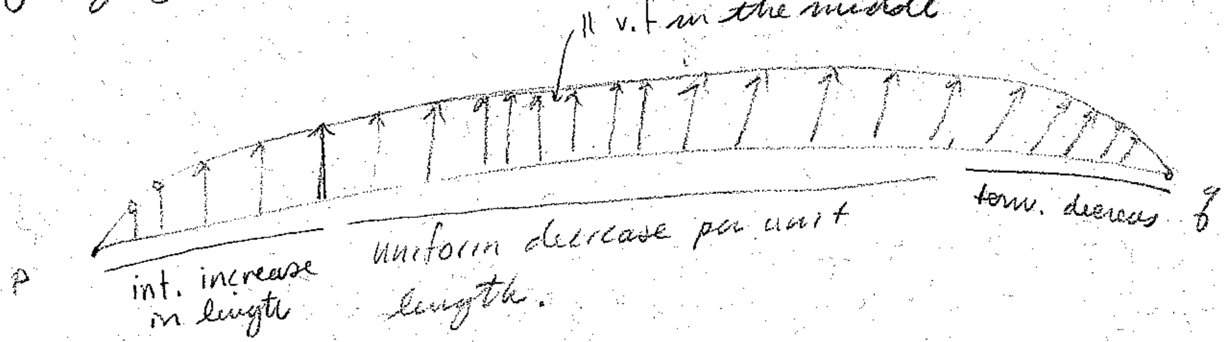
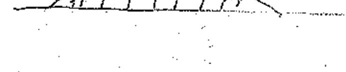
Idea of pf of Meyers' Thm — $c$ minimal geodesic, want to bound len by $\pi C^{-1/2}$



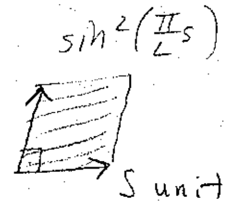

. increasing bits should be unif bounded. (no worse than Euclidean )
. middle bit can be as long as we want.

Pf: Let $c:[0,L]\to M$ be a minimal unit speed geod joining $p$ to $q$.
Need to show $L \le \pi C^{-1/2}$. Let $X$ be a unit len ∥ v.f.
along $c$. Set $T(s) = \sin\left(\frac{\pi s}{L}\right) X(s)$ at $c(s)$.



Then

$$\frac{1}{2}\frac{d^2}{dt^2}E(c_t)\Big|_{t=0} = \int_{c_0} |\overbrace{D_S T}^{=\frac{\pi}{L}\cos\left(\frac{\pi s}{L}\right)X}|^2 - \underbrace{R(T,S,T,S)}_{= K(S,T)\,|S\wedge T|^2}\, ds$$




$$= \int \frac{\pi^2}{L^2}\cos^2\left(\frac{\pi s}{L}\right) - \sin^2\left(\frac{\pi}{L}\right) K(S,T)\, ds$$

$$\le \int \frac{\pi^2}{L^2}\cos^2\left(\frac{\pi s}{L}\right) - \sin^2\left(\frac{\pi s}{L}\right) C\, ds = \frac{\pi}{2}\left(\frac{\pi^2}{L^2} - C\right)$$

By minimity $0 \le \frac{d^2}{dt^2}E(c_t)\Big|_{t=0}$ so $C \le \frac{\pi^2}{L^2} \Rightarrow L \le \pi C^{-1/2}$ as claimed. ∎

· Can strengthen to use Ric curv, see GHL 3.85.

[Cor: Let $G$ be a Lie group s.t. $\mathfrak{g}$ has trivial center. (if $X \in \mathfrak{g}$ has $[X,Y]=0$ for all $Y$ then $X=0$)
Then if $G$ has a biinvariant metric, then $G$ is compact] assign as HW?

issue: why is bound uniform

[Cor: $SL_n\mathbb{R}$ has no biinv metric]

Sphere Theorem: (1961) Let $M$ be a compact, simply connected Riemannian manifold whose sec curv sat $1/4 < K \leq 1$. Then $M$ is homeomorphic to $S^n$.

Notes: If $K > 0$ everywhere, can rescale so $\max(K) = 1$, so applies in general.

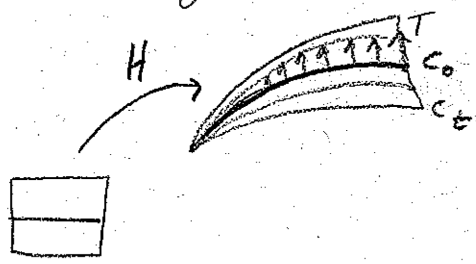
· In even dims, $\mathbb{CP}^n$ has a metric w/ $1/4 \leq K \leq 1$ so is sharp in this case

· In dim 2: $K > 0 \Rightarrow M \cong S^2$ by Gauß-Bonnet $\int K\, dA = 2\pi\chi(M)$
In dim 3: $K > 0 \Rightarrow M \cong S^3$ by Hamilton using Ricci flow.

Question: Does $S^2 \times S^2$ have a metric w/ $K(P) > 0$.

Jacobi Fields: [used to understand families of geod, exp. map.] 

$c: [0,L] \to M$ a geodesic, $H: [0,L]\times(-\varepsilon,\varepsilon) \to M$ be a var s.t. each $c_t$ also a geodesic. Set $S = H_*(\frac{\partial}{\partial s})$, $T = H_*(\frac{\partial}{\partial t})$. [What can we say about $T$ along $c_0$?]. Now $D_S S = 0$ everywhere

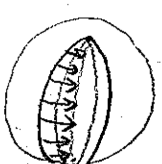



$$0 = D_T D_S S = R(S,T)S + D_S D_T S$$
$$= R(S,T)S + D_S D_S T$$

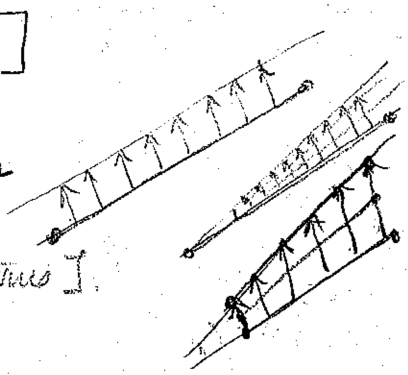
Def: $c$ a geod. A Jacobi field on $c$ is a v.f. $J$ sats:

$$J'' + R(c',J)c' = 0 \text{ where } J'' = \frac{D^2}{ds^2}J \text{ [is the } 2^{nd} \text{ der of } J \text{ along } c]$$

[will show every Jacobi field comes from a var of geod.]

Ex:  On $\mathbb{R}^2$, cond is $J''=0$, get $J' = $ const

[JUMP TO ✳] $\Rightarrow J = J_0 + sJ_1$ [Same on flat torus].



Thm: $c:[0,L]\to M$ a geod. Given $u,v \in T_{c(0)}M$, $\exists!$ Jacobi field $J$ along $c$ with $J'(0) = u$ and $J''(0) = v$. In particular, the space of all Jacobi fields is dim $2n$.

Pf: Let $X_i$ be $\parallel$ vector fields along $c$ which are an orthonormal basis at each point. Set $J = \sum a_i X_i$ for $a_i: [0,L]\to\mathbb{R}$. Then the Jacobi condition becomes

$$a_i'' = \sum_{i,j} a_i \underbrace{R(c',X_i,c',X_j)}_{\text{smooth fn of } s}$$

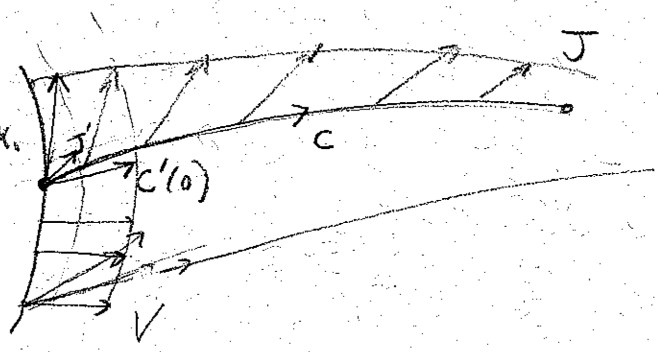
as $J' = \sum a_i' X_i$, $J'' = \sum a_i'' X_i$.

Theorem now follows from exist. and uniqueness of $2^{nd}$ order linear ODE's. ∎

Every Jacobi field comes from a geod variation: $\boxed{\text{mention exp case 1st}}$

Pick $\alpha:(-\varepsilon,\varepsilon)\to M$ a curve w/ $\alpha(0)=c(0)$

$\alpha'(0)=J(0)$. Let $V$ be $\parallel$ tran of $c'(0)$ along $\alpha$.




Let $Y$ be $\parallel$ trans of $J'(0)$ along $\alpha$.

at $\alpha(t)$ set

$$X(t)=V(t)+tY(t)$$

and make

$$H(s,t)=\exp_{\alpha(t)} sX(t):[0,L]\times(-\varepsilon,\varepsilon)\longrightarrow M$$

Let $T=H_*\left(\frac{\partial}{\partial t}\right)$. Then $T(0,0)=\alpha'(0)=J(0)$. What is $D_S T(0,0)$?

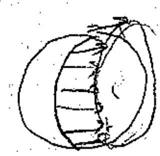
as $D_S T=D_T S$, consider $D_T S$ along $\alpha$. Then $S=X$, so

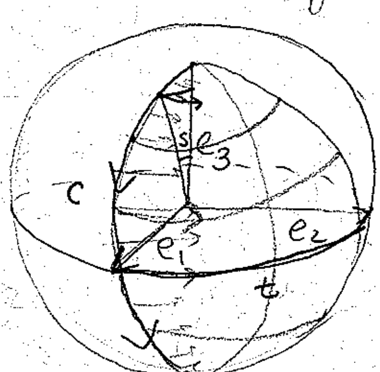
$D_T S=D_{\alpha'}X=Y(t)$. So $T'(0,0)=Y(0)=J'(0)$.

The infinite variation of $H$ is a Jacobi field $T$ w/ $T(0)=J(0)$ and $T'(0)=J'(0)$. By uniqueness part of last thm, have $T=J$ along all of $c$. So $H$ is the reg. geod. variation. ∎

[Var need not be unique, use exp map to see].

(☆) lule curve of $S^2$: $c:[0,\pi]\longrightarrow S^2$ unit speed.




$H$: geod var

$X$: $\parallel$ vector field shown (unit length)

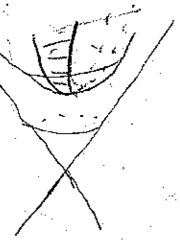
$J=\sin(s)\,X(s)$

$J'=\cos(s)\,X(s)$

$J''=-\sin(s)\,X(s)$

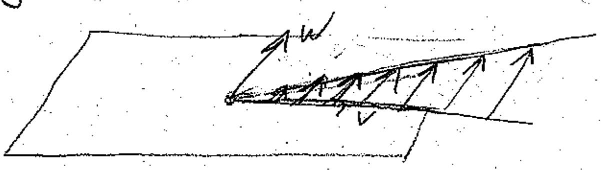
Jacobi $\Rightarrow g(J'',J)=-R(c',J,c',J)$

$-\sin^2(s)=-\sin^2(s)\,R(c',X,c',X)\Rightarrow K=1.$

Also works for $S^n$, $\mathbb{H}^n$  $= -1$. 

[Flat things here]

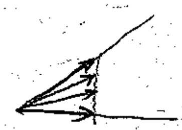
Example:




$$H(s,t) = \exp_p(s(v+tw))$$

$J = D\exp$ (inf var in $T_pM$) so

$J(0) = 0;$ $\qquad J(s) = D_{sv}\exp_p(sw)$.

$J'(0)$; $D_S T = D_T S$ and $H(0,t) = p$ so no connection is needed! 

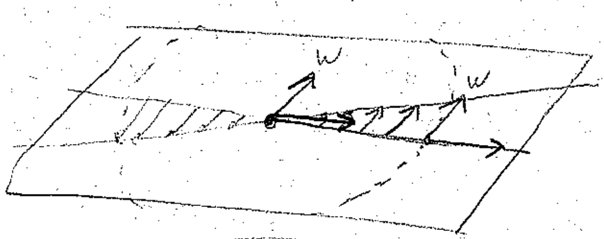
Thus $J'(0) = w$. Key

[RETURN TO JUMP]

[To show: connection to Gauß curvature]

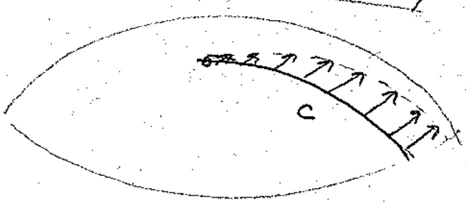
How does $d\exp$ extort dist? $v, w$ orthonormal unit vec in $T_pM$

$C = \exp(sv)$ a geod




$J = D_{sv}\exp(sw)$ Jacobi field.

$f(s) = \|J\|^2$ - smooth fn of $s$. (extend v.f. on $T_pM$)




Taylor: $f(0) = 0$, $f'(0) = 0$: $S\, g(J,J) = 2g(\underset{D_S J}{\overset{\shortparallel}{J'}}, J)$

$f''$: $S^2 g(J,J) = 2g(J'', J) + 2g(J', J')$; at $0 = 2\|w\|^2 = 2$

$f'''$: $= 2g(J''', J) + 6g(J'', J')$; at $0 = 0$

$-R(c', J)c' = 0$ at $0$.

$$J''' = D_s J'' = -D_s\left(R(c', J, c')\right)$$

$$= -\Big[(D_s R)(c', J, c') + R(\overset{0}{\cancel{D_s c'}}, J, c') + R(c', D_s J, c') + R(c', J, \underset{0}{\cancel{D_s c'}})\Big]$$

At $t=0$ $\quad = -R(c', J')c' = -R(v,w)v$

__*Finally:*__

$$f^{(4)} = 2g(\overset{\to 0}{\cancel{J^{(4)}, J}}) + 8g(J^{(3)}, J') + 6g(\overset{\to 0}{\cancel{J'', J''}})$$

At $0$:

$$= -8\,R(v,w,v,w) = -8\,K(v,w)$$

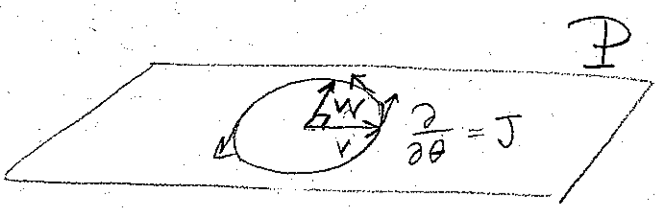
So

$$\text{(†)} \quad \|J(s)\|^2 = r^2 - \frac{1}{3}K(v,w)r^4 + O(r^5)$$

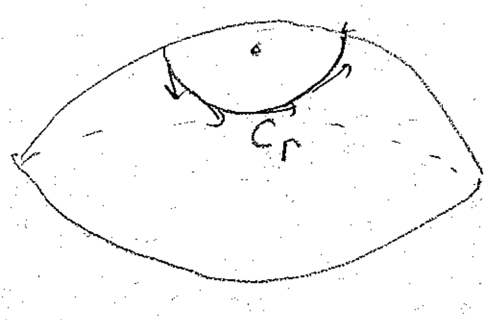
Hence

$$\text{(✪)} \quad \|J(s)\| = r - \frac{1}{6}K(v,w)r^3 + O(r^4) \quad \text{(and is smooth fn of } r\text{)}$$

[*Therefore*: $K > 0$ geod conv, $K < 0$ geod diverge.]




$$c_r(\theta) = \exp_p\left(r(\cos\theta\, v + \sin\theta\, w)\right)$$




$$L(c_r) = \int_0^{2\pi} \|J(r,\theta)\|\, d\theta$$

$$= 2\pi r\left(1 - \frac{1}{6}K(v,w)r^2 + O(r^3)\right)$$

__*Technical Point:*__ How do we know error term of (✪) is "cont w.r.t $\theta$"?

That is, for each $\theta$, $\|J(s,\theta)\| = r - \frac{1}{6}K(v,w)r^3 + E(r,\theta)$

w/ $|E(r,\theta)| \leq C_\theta r^4$ for all $r$ small.

Issue: could $C_\theta$, "$r$ small" vary w/ $\theta$?

Eg: $f(x,y) = \begin{cases} xy^2/(x^2+y^4) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$  For each $\theta$, $f(r,\theta) = 0 + O(r)$

But $f(x,y)$ is not $0 + O(r)$. du fact $f$ is not continuous ($=1$ on $x = y^2$)

[here dir deriv. are not consist w/ any linear total derivative].

Now, $\|J(r,\theta)\|^2$ is smooth fn on nbhd of $p$, as $J(r,\theta) = \exp \frac{\partial}{\partial\theta}$
$\overset{\shortparallel}{f(x,y)}$ is smooth. $= \exp\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)$

Thus it has a Taylor series $0 + \frac{\partial f}{\partial x}\Big|_0 x + \frac{\partial f}{\partial y}\Big|_0 y + \cdots$

Plugging in various directions $F_\theta(r) = f(r\cos\theta, r\sin\theta)$ and comparing to $(+)$

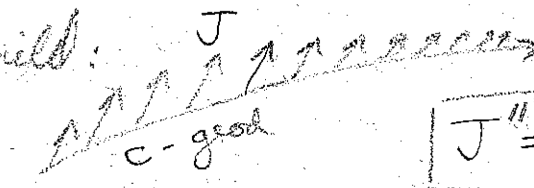
shows that

$$f(x,y) = r^2 - \frac{1}{3}K(v,w)r^4 + O(r^5) \quad \text{as a fn of two vars.}$$

(e.g. $f(x,y) = 0 + a_1 x + a_2 y + b_1 x^2 + b_2 xy + b_3 y^2 + c_1 x^3 + c_2 x^2 y + \cdots d_1 x^4 + \cdots + O(r^5)$)

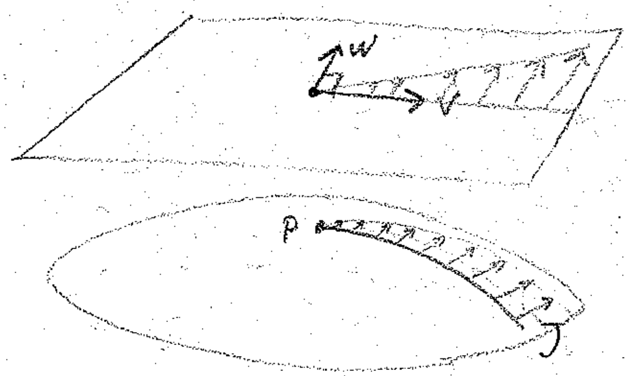
This completes the proof. ∎

# Lecture 12: Jacobi field:




$$\boxed{J'' = -R(c', J)c'}$$

## Connected to exp:




$c(s) = \exp_p(sv)$

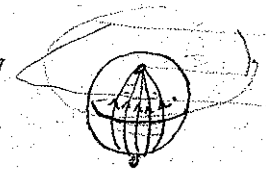
$J_w(s) = (D_{sv}\exp)(sw)$ - Jacobi field

Gauß lemma: $J_w \perp c'$ everywhere

$(D_{sv}\exp$ not an isom.$)$

$\iff$ exists $w \perp v$ s.t. $J_w(s) = 0$.

Ex: [Prompt]



**Def**: $c: [0, L] \to M$ a geod. Pts $p, q$ on $c$ are conjugate along $c$ if $\exists$ a nontrivial Jacobi field $J$ on $c$ vanishing at $p$ and $q$.

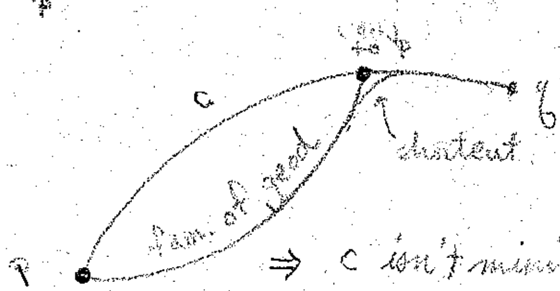
**Note**: 1) $\iff$ $D_v \exp_p$ singular where $\exp_p(sv): [0, 1] \to M$ traces out $c$.

2) By Gauß, $J \perp c'$ for such a $J$.

3) point out symmetry in

4) indep of param.

## Connection to minimal paths:



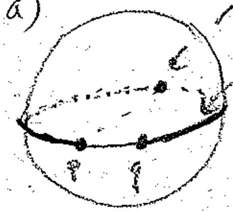

$\Rightarrow$ $c$ isn't minimal.

**Thm**: $c: [0, L] \to M$ a geod joining $p$ to $q$.

i) If $\exists$ a $s_0 \in (0, L)$ s.t. $p$ and $c(s_0)$ are conjugate, then $\exists$ a var $c_t$ of $c$ w/ fixed endpts s.t. $E(c_t) < E(c)$ for $t \neq 0$

ii) If $p$ has no conj pts on $c$, then $\nexists$ a var $c_t$ of $c$ w/ fixed end pts. s.t. $E(c_t) < E(c)$ for $t \neq 0$.

*Ex:* a)




conj pt

geod, yet can be shortened

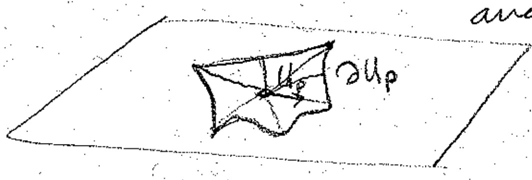
b) ∃ Flat torus: no conj pts. [prompt class]

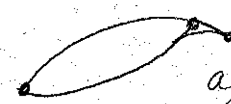
*Pf:* See GLH 3.73. Skip ahead, return at end, if time [wasn't]

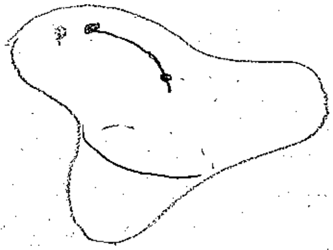
*Cut Locus:* $\boxed{M \text{ Complete}}$

Set $U_p = \{ v \in T_pM \mid c_v : [0, 1+\varepsilon] \to M \text{ is minimal geod for some } \varepsilon > 0 \}$

and $\mathrm{Cut}(p) = \exp_p(\partial U)$. So $M = \exp(U_p) \cup \mathrm{Cut}(p)$




$\mathrm{Exp}|_{U_p}$ is injective by  again.




By theorem, $D\exp$ is non-sing on $U_p$.

$\Rightarrow$ Combining, $\exp|_{U_p}$ is a diffeo.

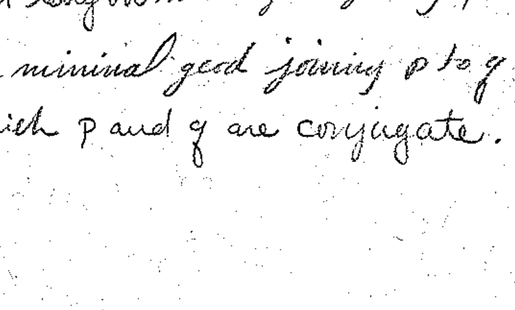
*Thm:* $q \in \mathrm{Cut}(p) \iff$ at least one of
a) ∃ distinct length min geod joining $p$ to $q$.
b) there is a minimal geod joining $p$ to $q$ along which $p$ and $q$ are conjugate.

*Pf:* HW. (See do Carmo, pg 268.)

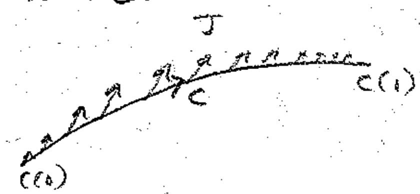
*Note:* $M \setminus \{p\}$ def retracts to $\mathrm{Cut}(p)$.

($S^2$, metric of quest origin) $\mathrm{Cut}(p)$ is contractible

is tree:
may not be finite if $g$ is not real analytic.



**Lemma** $M$ comp R-mfld with non-pos sectional curvature.
Then any geod $c$ has no conjugate points.



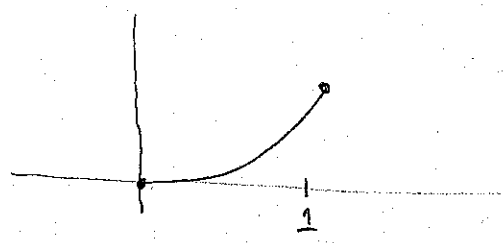

Pf. Let $J$ be a Jacobi field vanishing at $c(0)$ and $c(1)$.
Set

$$f(s) = g(J(s), J(s)) \geq 0$$

$$f'(s) = 2g(J', J)$$

$$f''(s) = 2g(J'', J) + 2g(J', J') = 2g(J', J') \underbrace{-2R(c', J, c', J)}_{\geq 0} \geq 0$$

$\Rightarrow f$ is convex up. $f(0) = f'(0) = 0$,
and $f(s) \geq 0$ everywhere. As $f(1) = 0$,
$f \equiv 0$ on $[0,1]$. $\Rightarrow J$ is trivial




Cartan-Hadamard: $M$ complete R-mfld with non pos sectional curve.
Then $\exp_p : T_pM \to M$ is a smooth covering map. In particular
a) If $M$ is simply connected, $M$ is diffeo to $\mathbb{R}^n$.
b) If $M$ is compact, then $\pi_1 M$ is infinite.

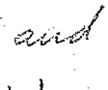
[Contrast with Myer's thm, under which $M$ is cpt and $\pi_1 M$ is finite]

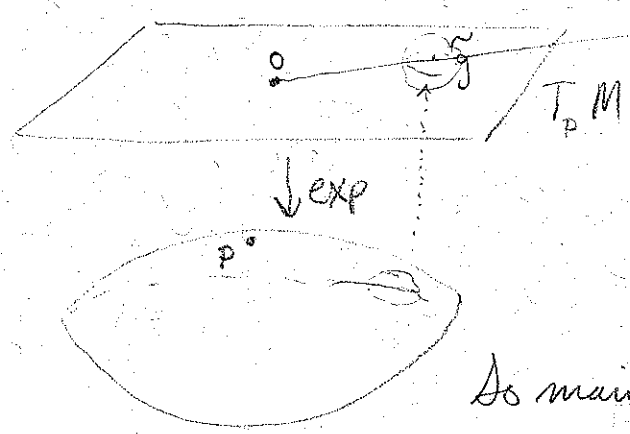
Appreciation: $\mathbb{R}^3 \setminus \{\vec{0}\}$ has comp metric $\left[\begin{matrix}\text{prompt}\\ \text{class}\end{matrix} \; S^2 \times \mathbb{R}\right]$

$S^3 \setminus \{1 \text{ dim cantor set}\}$

$= \text{Univ. cover of } L(3,1) \# L(3,1)$

For these $\pi_2 \neq 0$, but $\exists$ a contractible (open) 3-mfld,
$W$ the whitehead manifold, which is contractible.

Pf: By lemma and comp at the beginning, get $D\exp_p$ is non-sing everywhere. Why does it have to be a covering map? [prompt class for example:  ] Let $g$ be metric on $M$. Let $\tilde{g}$ be the pull back metric on $T_pM$; $(T_pM, \tilde{g}) \to (M, g)$ is a local isometry.




Claim: $(T_pM, \tilde{g})$ is complete.

Pf: $\exp_0$ is onto; geod are straight lines through $0$.

So main part of C-H follows from.

Lemma $(N, \tilde{g}) \xrightarrow{\pi} (M, g)$ a local diffeo and isometry.
If $(N, \tilde{g})$ is complete, then $N \to M$ is a covering map.

a) follows as a cover of simp. con spaces is 1-1.

b) If $\pi_1 M$ is finite; $\tilde{M}$ is cpt. By $\tilde{M} = \mathbb{R}^n$ by a)

Completes proof of C-H modulo $2^{nd}$ lemma. ▨

Pf of Lemma. Let $p \in M$, need to find a nbhd $U$ of $p$ s.t. $U$ is evenly covered $\left(\pi^{-1}(U) = \coprod V_\alpha \text{ w/ } V_\alpha \to U \text{ homeo}\right)$.

Choose $\varepsilon$ s.t. $\exp_p : B_\varepsilon(0) \to B_\varepsilon(p)$ is an embedding; set $U = B_\varepsilon(p)$ metric.

Let $\pi^{-1}(p) = q_i$. Let $V_i = B_\varepsilon(q_i)$. Since $\pi$ is a local isometry, have

$$(*) \quad \begin{tikzcd} T_{q_i}N \arrow[r, "\exp"] \arrow[d, "D_{q_i}\pi"'] & N \arrow[d, "\pi"] \\ T_pM \arrow[r, "\exp"] & M \end{tikzcd}$$

in particular, $M$ is complete.

Consider

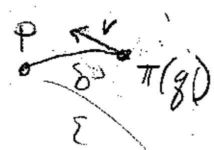
$$\begin{tikzcd} B_\varepsilon(0) \subseteq T_{q_i}N \arrow[r, "\exp"] \arrow[d, "\text{diff}", "\cong"'] & B_\varepsilon(q_i) \arrow[d, "\pi"] \\ B_\varepsilon(0) \subseteq T_pM \arrow[r, "\cong", "\text{diff}"'] & B_\varepsilon(p) \end{tikzcd}$$

Therefore, $\exp|_{B_\varepsilon(0)}$ in $T_{q_i}N$ is a diffeo as is $B_\varepsilon(q_i)$. So

$\bigcup V_i \subseteq \pi^{-1}(U)$ and $\pi|_{V_i}$ is a homeo. Still need:

1) $\pi^{-1}(U) = \bigcup V_i$ [ go back to  ]

2) $V_i \cap V_j = \emptyset$ for $i \neq j$

1) Suppose $q \in \pi^{-1}(U)$. Then $\exists$ a geod in $B_\varepsilon(p)$ joining 


Lift init tangent to $T_qN$, consider geod of length $\varepsilon$ w/ this tangent.
[exists by completeness]. By $(*)$, end pt of geod in $N$ must be one of $q_i$.
So $q \in V_i$.

This argument doesn't work

2) Suppose not. Then $\exists$ a path $\gamma$ in $\pi^{-1}(U)$ joining two distinct $q_i$.
But then $\pi \circ \gamma$ is a non-trivial elt of $\pi_1(M)$ contained in the contractible $U$, a contradiction. ∎

*Note*: Proof of CH uses very strongly the Hopf-Rinow theorem.

Lecture 13:

[Myers, Synge, Cartan-Hadamard, etc, but what are examples?] (30)

[Put Poincare, S-W on board ahead of time?]

$(M, g)$ w/ const sectional curve: rescale so that $= +1, 0, -1$

Ex: $+1$: $S^n$, $\mathbb{RP}^n$ $\leftarrow$ will show today this is all there is in even dim.

General Form: $S^n/\Gamma$ where $\Gamma$ is a finite subgroup of $\mathrm{Isom}(S^n) \cong O(n+1)$

Lens Spaces: (3-d, but same works all odd dim)

$$S^3 \subseteq \mathbb{C}^2 \qquad \text{pick } n, k \text{ w/ } \gcd(n,k), \ \zeta = e^{2\pi i/n}$$

$$S^3 = \{(z,w) \mid |z|^2 + |w|^2 = 1\} \qquad \Gamma = \langle (z,w) \longmapsto (\zeta z, \zeta^k w) \rangle \cong \mathbb{Z}/n\mathbb{Z}$$

$\Gamma$ acts fixed point free, subgroup of $O(4)$. $L(n,k) = S^3/\Gamma$.

Even in 3-d there are other examples: Poincare Homology sphere, $\pi_1 = SL_2 \mathbb{F}_5$.

Ex: $0$: $\mathbb{R}^n$, $T^n = \mathbb{R}^n/\mathbb{Z}^n$.

dim: 2: 2 cpt mflds: $T^2$, Klein bottle.
3: 10 cpt flat manifolds

$\Big\}$ Bieberbach: in any dim, a compact flat manifold is finitely covered by $T^n$!

Ex: $-1$: $\mathbb{H}^2$, will show other examples later, including cpt ones.

[Notice that there is only one simply connected guy for each curvature sign.]

**Thm:** $(M,g)$ complete R-mfld w/ const sect. curv $K = +1, 0, -1$

If $\pi_1 M = 1$ then $M$ is <u>isometric</u> to $\begin{cases} S^n & \text{if } K=1 \\ \mathbb{R}^n & \text{if } K=0 \\ \mathbb{H}^n & \text{if } K=-1. \end{cases}$

For any $M$, its universal cover $U$ is one of these.

**Note:** $M = U/\Gamma$ w/ $U = S^n, \mathbb{R}^n, \mathbb{H}^n$ and $\Gamma$ a discrete subgroup of $\mathrm{Isom}(U)$.

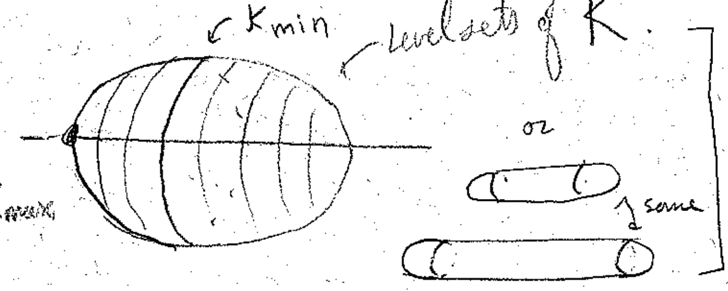
[compare to flat torus, Klein bottle]

**Contrast:** In gen, curv does not determine the metric.

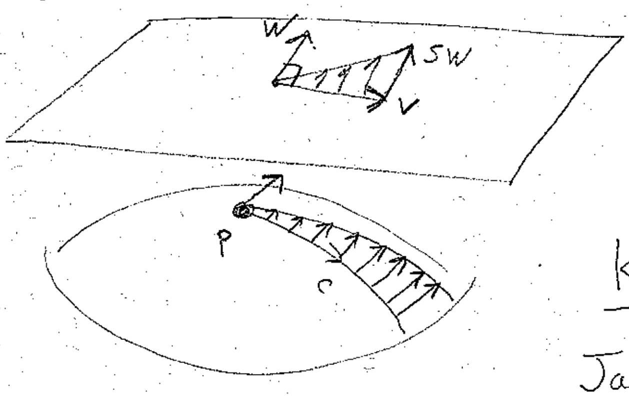
2-d: $K$ is just a fn, but $g = g_{ij}$ is 3 fns.

$\exists$ metrics $g_1, g_2$ on $S^2$ w/ $K_1, K_2$ same as fns but not isometric

[give surface of rev. examples.
Point out that $\pi_1 S^2 = 1$, and exp. why don't use flat torus as example]




**Pf:** $\pi_1(M) = 1$; [argue as in Cartan–Hadamard]




$v, w$ orthonormal.

$c(s) = \exp_p(sv)$

$J_w(s) = D_{sv} \exp_p(sw)$

<u>$K=0$:</u> [will show $\exp_p$ is an isometry]

Jacobi cond: $J'' = 0 \Rightarrow J'$ is parallel.

Let $W(s)$ be $\parallel$ tns of $w = J'(0)$. [$W = J'$]

Then $J(s) = sW(s)$. Thus $D_{sv}\exp_p(w) = W(s)$. So

$\|D_{sv}\exp_p(w)\| = \|W(s)\| = 1$. Combined w/ Gauß lemma, have $D_{sv}$ is an isometry.

Thus $\exp: T_pM \to M$ is a local isom. As $T_pM$ is complete (isom to $\mathbb{R}^n$), by lemma of last time $\exp_p$ is a covering map. Since $\pi_1 M = 1$, $\exp$ is a diffeo. As it is also an isom, $M$ is isometric to $T_pM$ and hence to $\mathbb{R}^n$.

$\underline{K=-1}$: [get diff as before but now not an isometry]

<u>Claim</u>: If $a$ and $b$ are orthonormal then $R(a,b)a = -b$,

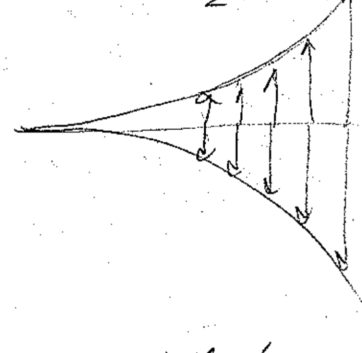
Take $c \perp$ to $a+b$, and note $2K(a, b+c) = R(a, b+c, a, b+c)$
$= R(a,b,a,b) + R(a,c,a,c) + 2R(a,b,a,c) = 2K + 2R(a,b,a,c)$
$\Rightarrow R(a,b,a,c) = 0$. So $R(a,b)a = R(a,b,a,b)\, b = -b$

Alt; note in const curv $R(a,b,c,d) = K\big(g(a,c)g(b,d) - g(b,c)\cdot g(a,d)\big)$

<u>Jacobi</u>: $J'' = -R(c', J)c' = J$. Again take $W(s)$ as before.

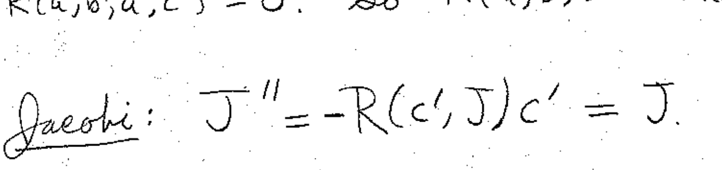
<u>Soln</u>: $J(s) = \sinh(s)\, W(s)$ where $\sinh(s) = \frac{1}{2}(e^s - e^{-s})$

since $J(0) = 0$, $J'(0) = \cosh(0) W(0) = w$.



So $\| D_{sv} \exp(w) \| = \frac{1}{s} \| J(s) \| = \frac{\sinh(s)}{s}$.

Let $\tilde{g}$ be pull back metric to $T_pM$, has radial form:




metric here same as Euclidean but scaled by $\frac{\sinh(s)}{s}$.

Also, $\exp: (T_pM, \tilde{g}) \to (M, g)$ is an isom by comp of the former and simple connectivity of the later.

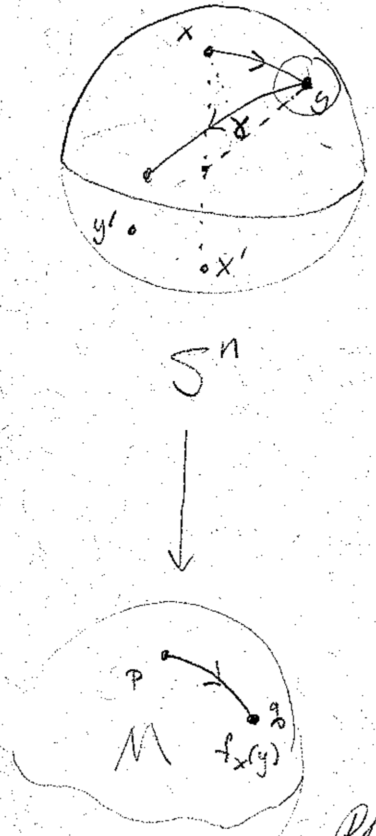
As $\tilde{g}$ only depended on const curv $-1$ any two such are isometric.

In particular pick an isom of inner prod sps $I : (T_{\bar{x}}\mathbb{H}^n, \text{std}) \to (T_pM, g_p)$ and then $\exp_p \circ I \circ \exp_{\bar{x}}^{-1} : \mathbb{H}^n \to M$ is an isometry.

$\underline{K = +1}$. [same as last case but now exp isn't a covering map]

Jacobi: $J'' = -R(c', J)c' = -J \Rightarrow J(s) = \sin(s)\, W(s)$

Thus: $\exp_p$ is a local diff on $B_\pi(0)$ with pull-back metric with radial form:


Let $I_x : T_xS^n \to T_pM$ be an isom. Then $f_x : S^n \setminus \{x'\} \to M$ given by $\exp_p \circ I_x \circ \exp_x^{-1}$ is a local isometry. Set $q = f_x(y)$ and set $f_y : S^n \setminus \{y'\} \to M$ by $\exp_q \circ D_y f_x \circ \exp_y^{-1}$.

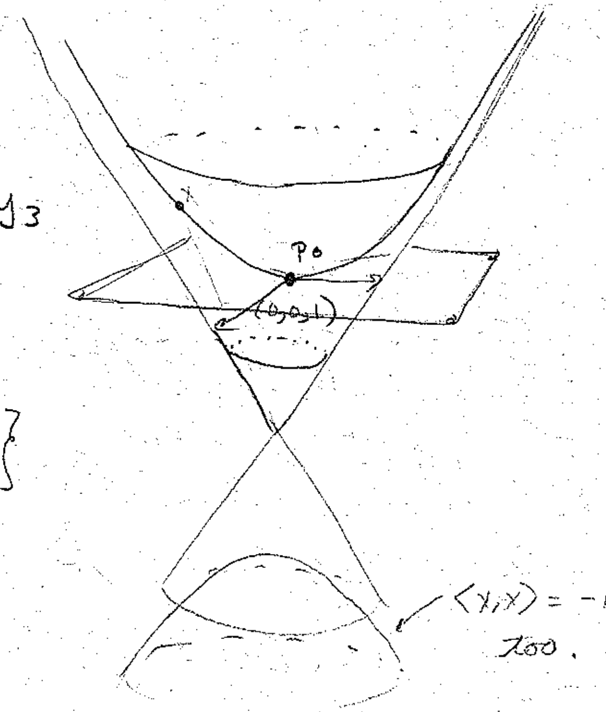
Claim $f_x = f_y$ on $S^n \setminus \{x', y'\}$

If so, set $f : S^n \to M$ by $f = f_x$ or $f_y$. Then $f$ is a local isometry. As $S^n$ is complete, $f$ is a covering map. As $\pi_1 M = 1$, $f$ is an isometry.

Pf of claim: By def of $f_y$ have $f_x(y) = f_y(y) = q$ and $D_y f_x = D_y f_y$. As local isometries take geodesics to geodesics, if $\gamma$ is a geod $\subseteq S^n \setminus \{x', y'\}$ starting at $y$, then $f_x \circ \gamma = f_y \circ \gamma$ since $D_y f_x(\gamma'(0)) = D_y f_y(\gamma'(0))$. Thus $f_x = f_y$ on $S^n \setminus \{$min geod joining $x'$ to $y'\}$. Switching to some $z \in S^n \setminus \{\text{geod } x' \to y'\}$ off the great circle containing $x$ and $y$ finishes the proof. // $\blacksquare$

**Hyperbolic Geometry:** [will concentrate on 2-dimensions]

~ A tale of 3-models ~



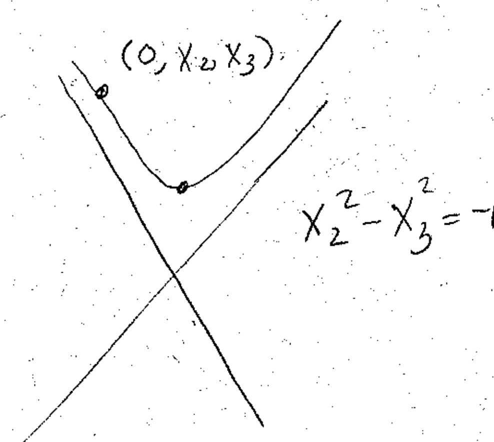

**Hyperboloid:** $x, y \in \mathbb{R}^3$, $\langle x, y\rangle = x_1y_1 + x_2y_2 - x_3y_3$

$$\mathbb{H}^2 = \{\langle x,x\rangle = -1,\ x_3 > 0\}$$

$$O(2,1) = \{A \in GL_3(\mathbb{R}) \mid \langle Ax, Ay\rangle = \langle x,y\rangle\}$$

$O_0(2,1)$ those which preserve $\mathbb{H}^2$.
$SO_0(2,1)$ those w/ det 1 (equiv, pres orient).
Claim: $SO_0(2,1)$ acts transitively on $\mathbb{H}^2$

- Any $M \in O(2)$ gives such an element via $\begin{pmatrix} M & \begin{matrix}0\\0\end{matrix} \\ \begin{matrix}0 & 0\end{matrix} & 1\end{pmatrix}$
- $\begin{pmatrix} 1 & 0 & 0 \\ 0 & x_3 & x_2 \\ 0 & x_2 & x_3 \end{pmatrix}$ takes $(0,0,1)$ to $(0, x_2, x_3)$




Now $\langle\ ,\ \rangle|_{T_p\mathbb{H}^2}$ is pos def for any $p$ by sym. as it is at $(0,0,1)$. Take $(\mathbb{H}^2, g = \text{res of } \langle\ ,\ \rangle)$

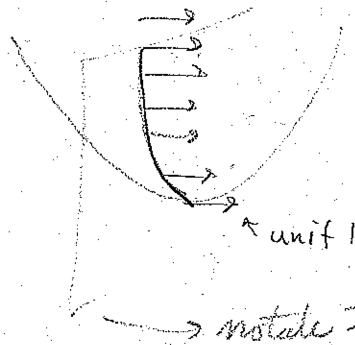
Then $\mathrm{Isom}(\mathbb{H}^2) = O_0(2,1)$ : $\supseteq$ is clear and $=$ true by transitivity of action on $UT\mathbb{H}^2$.

Homog $\Rightarrow$ const curvature $K$, complete

**Geodesics:** $\gamma = (\text{Plane through } \vec{0}) \cap \mathbb{H}^2$.

Pf: Things of this form inv under $O_0(2,1)$ so suffices to check for geod. through $p_0$. But there clear by using reflectional symmetry.

To see $K=-1$:

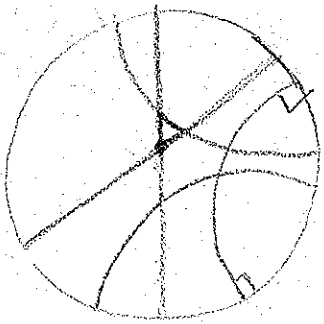



$J(s) = \sinh(s)\, W(s)$

## Lecture 14:

[More on hyperbolic geometry.]

**Poincaré Model:** $D = \{ z \in \mathbb{C} \mid |z| < 1 \}$, $g = \frac{4}{(1-r^2)^2} g_{Euclid}$.



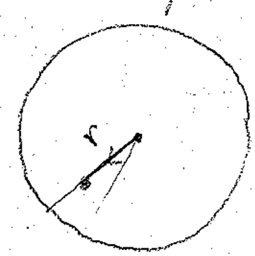
angles are what they look like

· Will see, const curve $-1$, geodesics are circles meeting $\partial D$ in right angles

[Emphasize conformality here]

Basic observations:

1) • By reflection symmetry, lines through $0$ are paths of geodesics



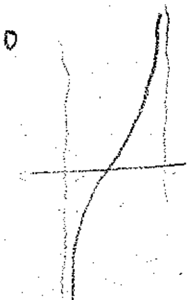

• $(D, g)$ is complete:

If $|z| = r$ have $d_g(0,z) = \int_0^r \frac{2}{(1-t^2)}\, dt$

$$= \int_0^r \frac{2}{(1-t)(1+t)}\, dt \quad \left[\text{blows up like } -\ln(1-r) \text{ as } r \to 1\right]$$

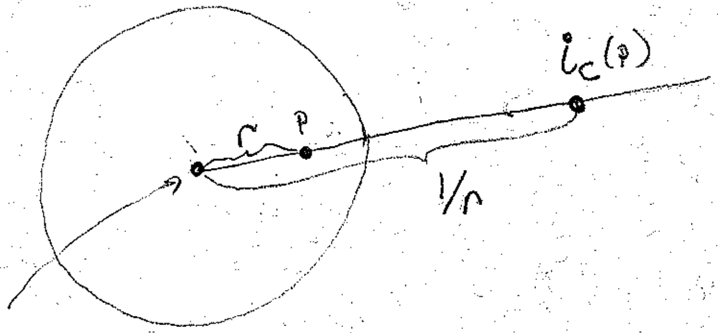
$$= \int_0^r \frac{1}{1-t} + \frac{1}{1+t}\, dt = -\ln(1-t) + \ln(1+t) \Big|_0^r$$

$$= \ln \frac{1+r}{1-r} = \tanh^{-1}(r)$$



[What are other geodesics? find other "reflections" in $\mathrm{Isom}(D,g)$.]

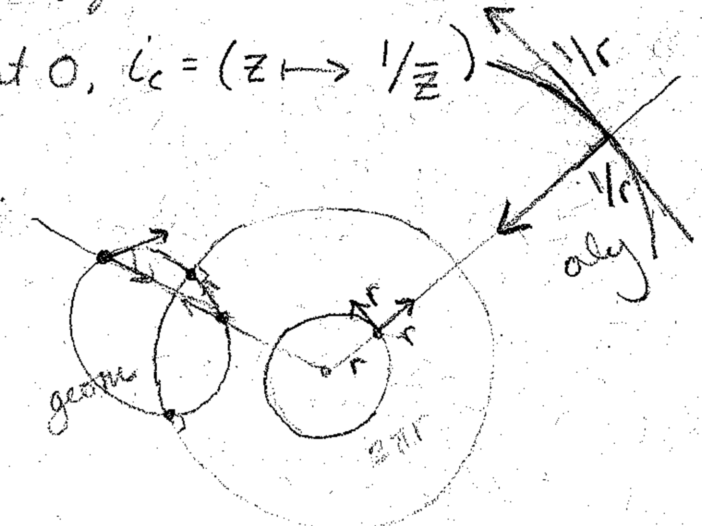
C circle in $\mathbb{C}$. $i_C$ = inversion in $C$ : $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ ; $\hat{\mathbb{C}} = \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$




Prop:
- $i_C \circ i_C = id$
- circles not containing center of C go to circles not containing center.
- circle going through center go to lines
- lines go to circles or lines.

center $\longleftrightarrow \infty$

Not hard to check alg.: do case of unit circle about 0, $i_C = (z \mapsto 1/\bar{z})$

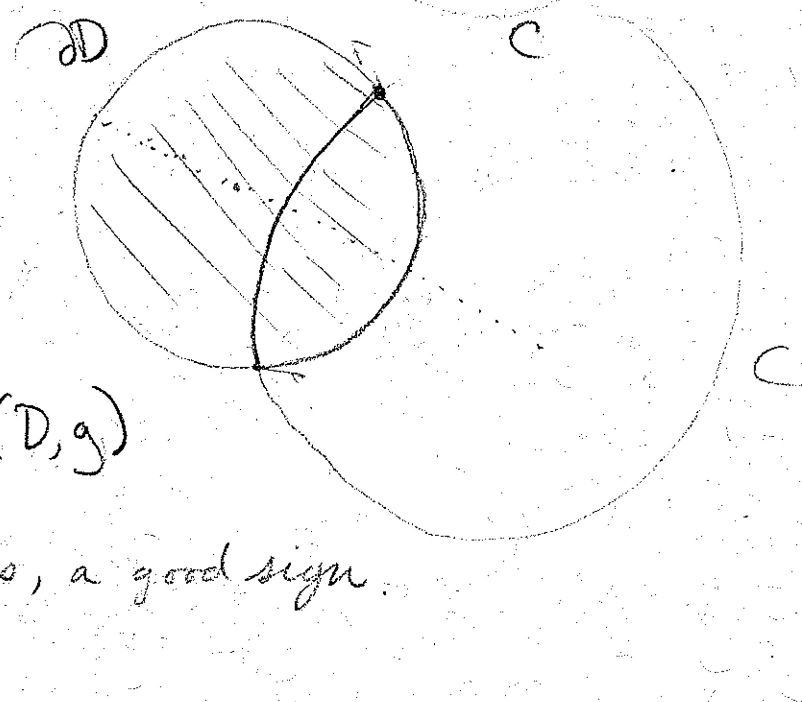



$i_C$ preserve angles (is conformal)
- alg arg. first.
- then geom, noting circles $\perp$ to C are preserved

Back to Poincaré model:

C $\perp$ to $\partial D$

$i_C$ pres D.




Claim: $i_C$ is an isometry of $(D, g)$

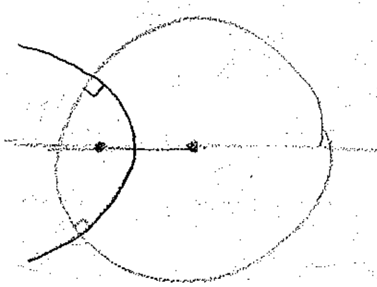
Plane argument: $i_C$ pres. angles, a good sign.

Arg from Thurston's book  ← must be equidist curve. ]

Conseq: 1) geod are circles meeting D in right angles.

2) $(D, g)$ is homogeneous.

$\Rightarrow$ curvature is const, check is $-1$.



$\Rightarrow$ via last time that $D \cong \mathbb{H}^2$.

Also: $\mathrm{Isom}(D)$ is gen by inversions.

Fact: $\mathrm{Isom}^+(D) =$ biholomorphic maps of D to itself

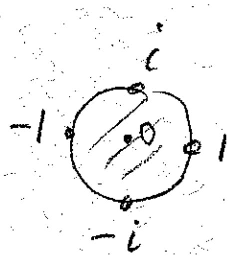
[ pf: as Isom(D) pres angles, note comp of prod of two inversions is clearly biholomorphic ]

Alt model: $H = \{ z \in \mathbb{C} \mid Re(z) > 0 \}$

$$g = \frac{1}{y^2} g_{Euclid}$$

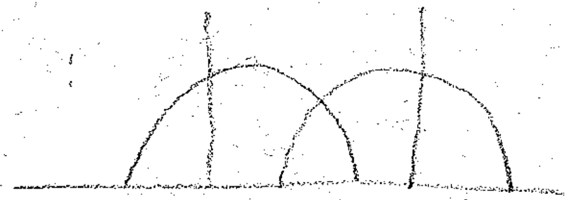

$D \longrightarrow H$ via

$z \longmapsto \frac{z+i}{iz+1}$




pres metric

Geod:



Isom(H): gen by inversions.

orient pres: $z \longrightarrow \frac{az+b}{cz+d}$  $a,b,c,d \in \mathbb{R}$  Möbius Trans.

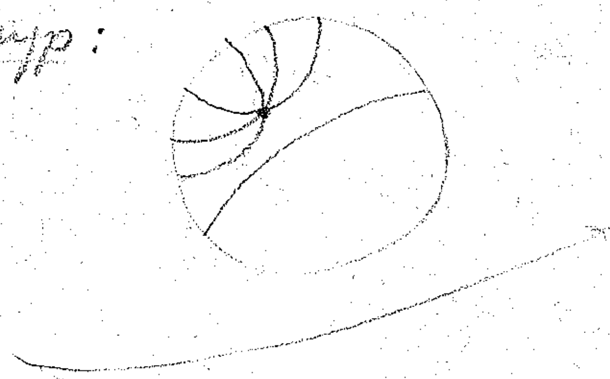
rescale, etc.

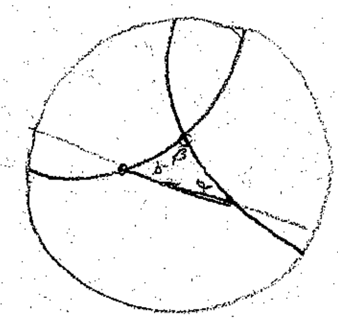
$$PSL_2\mathbb{R} \longrightarrow \mathrm{Isom}^+(H) \quad \text{a } \underline{\text{gp}} \text{ isomorphism.}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

---

Violates // hyp:

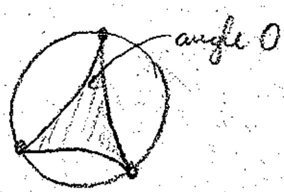





Triangles:

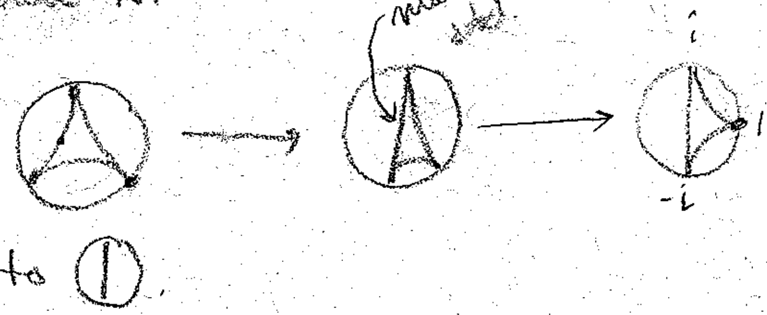
$$\mathrm{Area}(\triangle) = \pi - (\alpha+\beta+\gamma) \qquad \left(\text{sphere } (\alpha+\beta+\gamma) - \pi\right)$$

Ideal triangle:
(vert "at $\infty$")

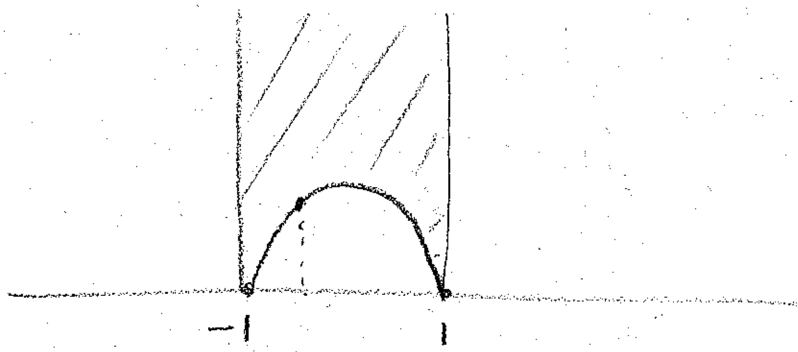



Lemma: any ideal $\triangle$ has area $\pi$.

Pf: Any two are isometric via inverting in circles $\perp$ to (I).



now in $\mathbb{H}$.



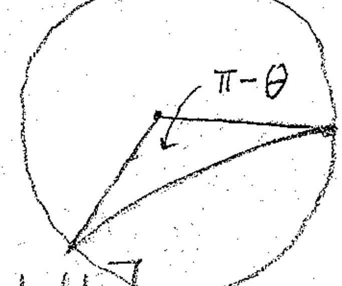

$$\int_{-1}^{1}\int_{\sqrt{1-x^2}}^{\infty}\frac{1}{y^2}\,dy\,dx$$

$$= \int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}\,dx \overset{x=\sin\theta}{=} \int_{-\pi/2}^{\pi/2} 1\,d\theta = \pi. \qquad \blacksquare$$

<u>Rmk</u>: only true for $\Delta$'s, ideal squares have moduli

$\mathrm{Isom}^+(\mathbb{H}) = PSL_2\mathbb{R}$ acts on $\hat{\mathbb{R}} = \mathbb{R}\cup\{\infty\}$ via Möb/proj trans. is det. by what it does to 3-pts.

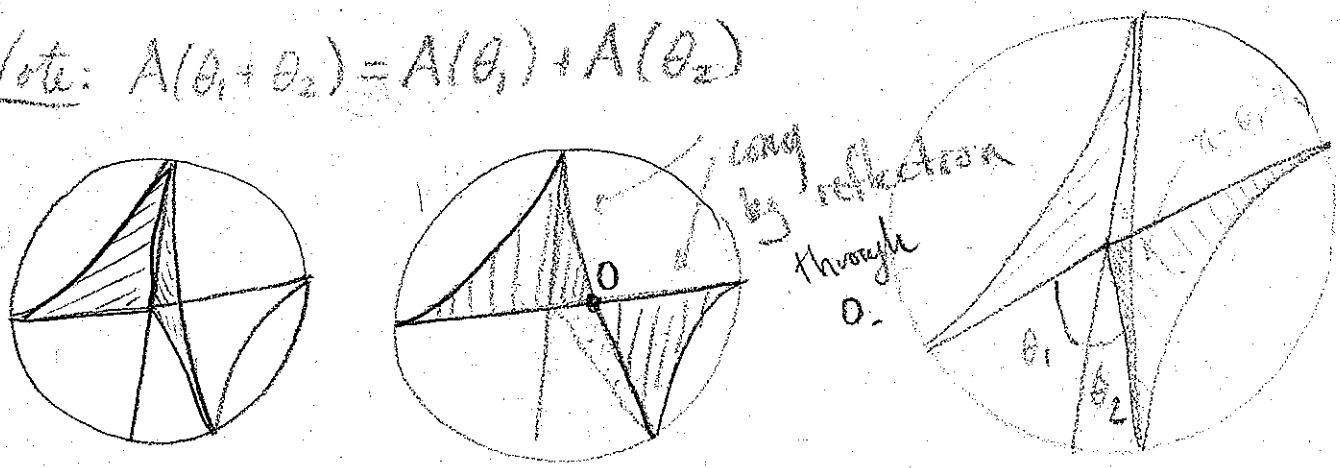
For each $\theta\in(0,\pi)$ $\exists!$ $\frac{2}{3}$-ideal tria.:




Let

$A(\theta)$ = Area of this $\Delta$ [$=\theta$ if form holds]

<u>Note</u>: $A(\theta_1+\theta_2) = A(\theta_1)+A(\theta_2)$

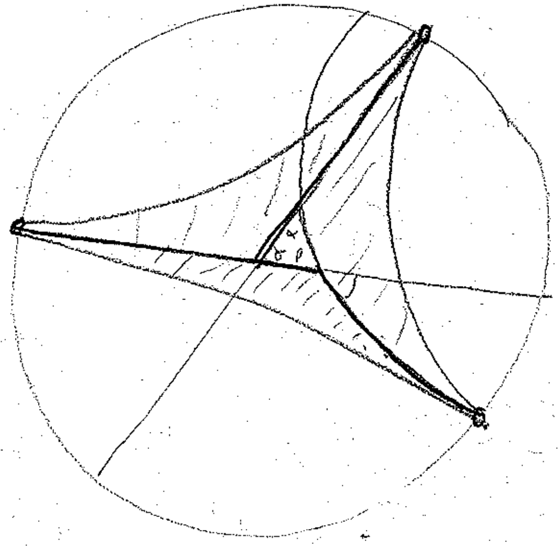



$\Rightarrow A$ is $\mathbb{Q}$-linear $A(\frac{p}{q}\theta_1) = \frac{p}{q}A(\theta_1)$

$\Rightarrow$ by cont, $A$ is $\mathbb{R}$-linear

$\Rightarrow A(\pi)=\pi \Rightarrow A(\theta)=\theta$ $\blacksquare$

In general

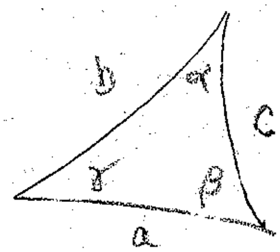


Area $\Delta(\alpha,\beta,\gamma)$

$= \pi - A(\beta) - A(\alpha) - A(\gamma)$

$= \pi - \alpha - \beta - \gamma.$

[Generalizes to polygons: $\Sigma$(int $\angle$) < Euclid case,
defect is area]




Hyperbolic trigonometry:

$$\frac{\sinh(a)}{\sin\alpha} = \frac{\sinh(b)}{\sin\beta} = \frac{\sinh(c)}{\sin\gamma}$$

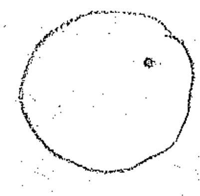
Pyth. theorem: $\gamma = \pi/2$ : $\cosh c = \cosh(a)\cosh(b)$

---

Class of isometries: $\sigma \in \mathrm{Isom}(D)$; $\sigma$ extends to $\partial D$.

Must have f.p. $p$ by Brouwer.

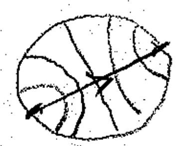
Elliptic: $p \in D$ acts as rotation



$\sigma \leftrightarrow A \in PSL_2\mathbb{R}$; case eigenvalues imagin

Parabolic: unique $p$, which is in $\partial D$: show in $\mathbb{H}$, then $D$
eigenvalue 1

Hyperbolic: two f.p. on $\partial D$, eigenvalues real

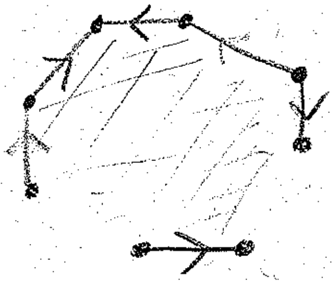


More to after examples of surfaces

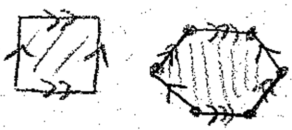
Lecture 15: Hyperbolic geometry, the conclusion. [Draw fig up before class starts]

[So far, talked about the hyp. plane in and of itself; today will focus on compact Riemannian mflds w/ const $-1$ curvature.]

Ex: $M =$ 



top. pict

As in flat case build metric out of geod polygon: 

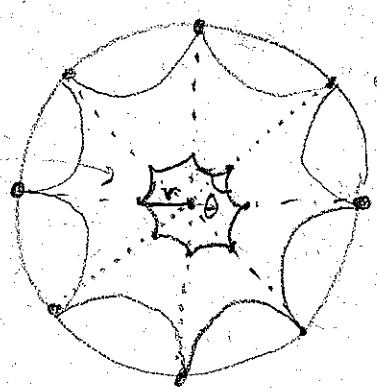
Need: glued edges have same length.

[Prompt]: angles around vert $= 2\pi$

Need: reg hyp octogon w/ int angle $\pi/4$ [compare w/ trying to put a Euclid metric on it]

Existence by cont

regular 8-gon of "radius" $r$



← evenly spaced pts

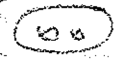
When $r$ very small $\theta$ is nearly $\frac{3}{4}\pi$

When $r$ very small $\theta$ nearly $0$.

Thus $\exists$ desired polygon, and metric on  of const curv. $-1$.

Consider univ cover $(\mathbb{H}^2, \text{std metric})$

$\downarrow$

$($$, g)$

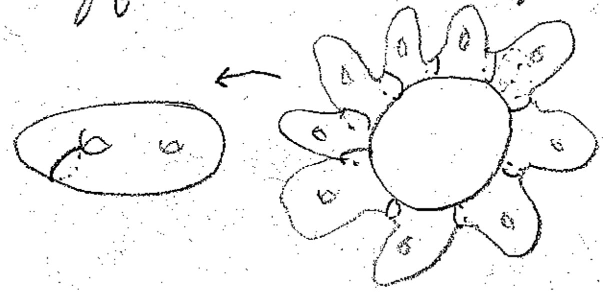
Get tiling of $\mathbb{H}^2$ by octogons. Show class.

· Discuss case of pentagons.

· Same construction gives metric on any orient surface.

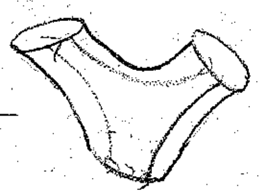
Metric is not unique, just as with tori.

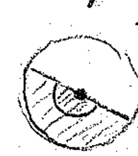
Diff metrics on surfaces of genus 10: $\begin{cases} \cdot \text{ regular 40-gon} \\ \cdot \text{ 9 fold cover of } ((\text{oo}), g) \end{cases}$

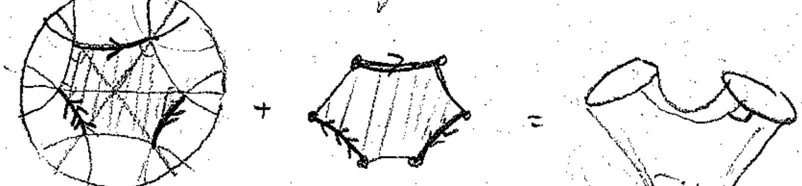


Dist by largest $\delta$ s.t $\exists$ some pt where $\exp_p$ is an emb on $B_\delta(\vec{0})$.

How do we param all hyp. structures on a surface?

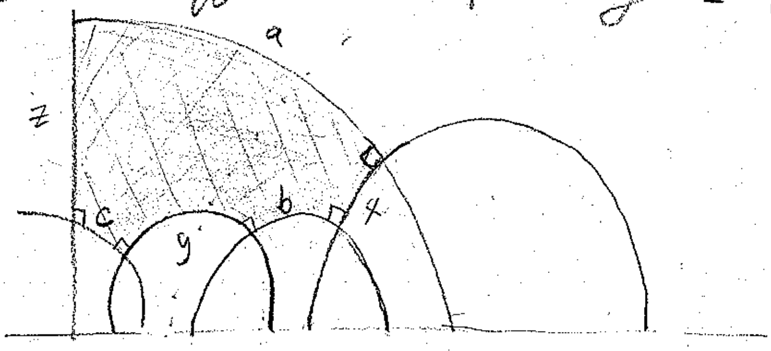
<u>Pairs of Pants</u>: $P =$  ← where has a hyp. metric w/ geodesic boundary.

i.e. nhlds of $\partial$ pts are <u>isom</u> to nhlds of geod half spaces 

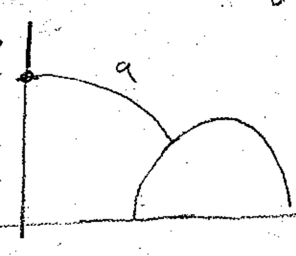

<u>Example</u>: 2 reg ⊾ hexagons 

Can specify the 3 cuff lens independently: [maybe easiest to do some trig]

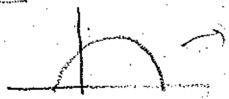
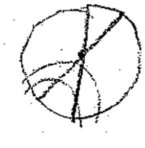
cln $H$

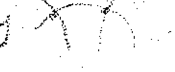



Start w/ a edge:




Here a is the unique min geod joining these two ⊥ and ⌒

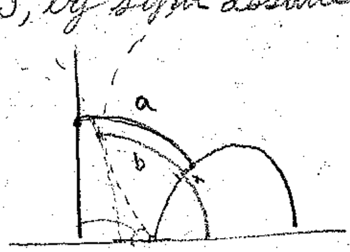
<u>Aside</u>: Note why can't have  

ultra ‖ geod joined by unique min geo 

can also have 

Now, by sym assume $a \geq b$.

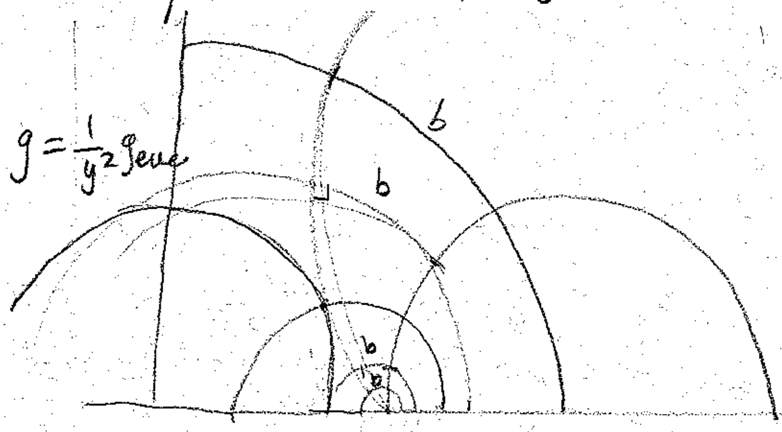
For small x have and no room for a hex.



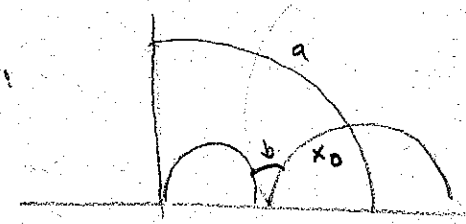
What path does endpt of b take? Recall from last time pts of equidist from a geod lie on a circle

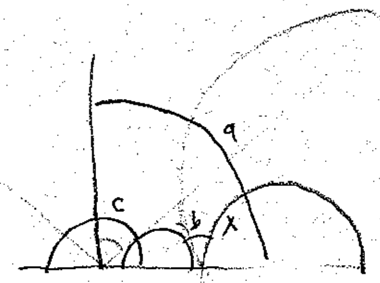
Next side is tangent to blue.

$g = \frac{1}{y^2} g_{euc}$



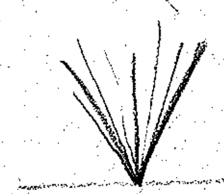
See $\exists\, x_0$ w/ this pick!





For $x > x_0$, $\exists!$ hex w/ sides $(a, x, b)$

Look at equidist curves from $|$ :



From this, its clear that $c(x)$ is a monotone increasing fn on $(x_0, \infty)$ w/ $\lim_{x \to x_0} c(x) = 0$ and $\lim_{x \to \infty} c(x) = \infty$.

<u>Thm</u>: Given $a, b, c > 0$, $\exists$ right angle hex w/ sides

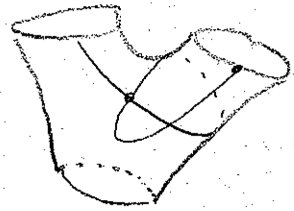
which is unique up to $\mathrm{Isom}^+(\mathbb{H}^2)$.

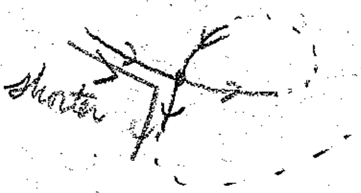
<u>Cor.</u> Given $a, b, c > 0$, $\exists$ a hyperbolic pair of pants w/ cuff lengths $a, b, c$.

<u>Claim</u>: In fact, there exists a unique such pair of pants.

Choose a min geod $\gamma$ joining a pair of cuffs 

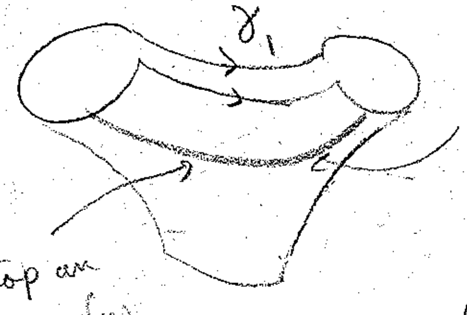
Know: · $\gamma$ meets $\partial$ in right angles.

· $\gamma$ is embedded:  look at 1st intersection



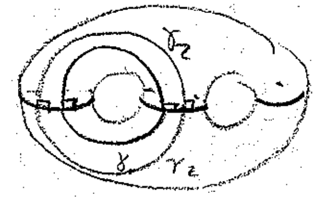

· $\gamma$ is unique: suppose $\gamma_1$ + $\gamma_2$ are two min geod. Must be disjoint: 


$len(\alpha_1) + len(\alpha_2) < len(\gamma_1) + len(\gamma_2) = 2\, len(\gamma_1)$
$\Rightarrow \gamma_i$ are not minimal



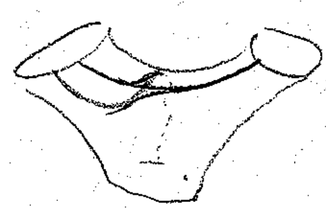

$\gamma_2$ must be this curve, or sim one on back

top an annulus 

Now double P to get 

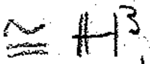
← has −1 metric

w/ pair of disjoint closed geod. From these are homotopic and so by HW must have $\gamma_1 = \gamma_2$.

Now let $\gamma_1, \gamma_2$, and $\gamma_3$ be these min geod joining each pair of cusps. The $\gamma_i$ can't intersect: have some common cuff:  Now do exchange arguement.

$len(\alpha_1) + len(\alpha_2) < len(\gamma_1) + len(\gamma_2)$

So: 


Double to get , pass to univ cover $\tilde{M} \cong \mathbb{H}^3$, lift to get $\tilde{f}$ which is a local isom embedding.

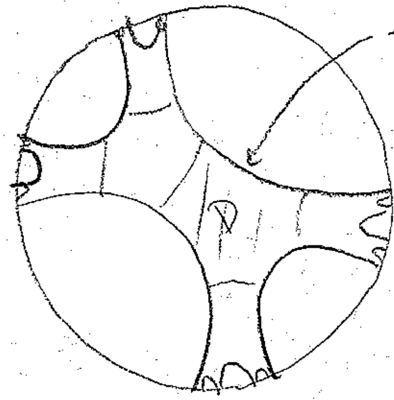
$$\begin{array}{ccc} & \overset{\tilde{f}}{\dashrightarrow} & \tilde{M} \\ & & \downarrow \\ D \hookrightarrow P & \hookrightarrow & M \end{array}$$

D is right angle hex, hence det by lens of $\gamma_i$.

$P \setminus \mathring{D}$ is also a right angle hex, must be isometric. So P comes from our construction. ∎

Aside: $\tilde{P} \hookrightarrow \mathbb{H}^2$

$\pi_1(P)$ = Free gp on two gens




Action of $\pi_1 P$ on $\tilde{P}$ extends to $\mathbb{H}^2$.

Thus have a free group $\subseteq \mathrm{Isom}(\mathbb{H}^2)$

[ If time, go back to class of isom discuss in this context or $\mathbb{H}^2/\pi_1(P)$ etc, or rough sketch of Teich space. ]

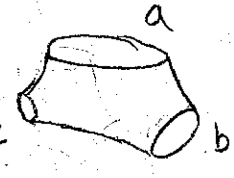
<u>Lecture 16</u>: Teichmüller Space: the space of all hyp. metrics on a fixed surface. ← called <u>hyp. metrics</u>

S clsd surface of genus $\geq 2$: $\mathcal{M}S = \left\{ g \mid g \text{ a metric on } S \text{ of const curv } -1 \right\} \Big/ g_1 \sim g_2$ if $\exists$ an isom $(S, g_1)$ to $(S, g_2)$

called the <u>moduli space</u> of S.

Last time consider hyp metric on  w/ tot. geod boundary.

What is $\mathcal{M}P$? Thm: Given $a, b, c > 0$, $\exists!$ hyp. metric on P up to isometry with cuff len




<u>Note</u>: Top, all cuff are the same, so this is isom to  which is a diff metric.



Thus $\mathcal{M}P = \mathbb{R}^3_+ / S_3$ $\left[ = \right.$  $\left. \times \mathbb{R}^+ \text{ top} \right]$

[Will adopt this strategy for closed surface case] 

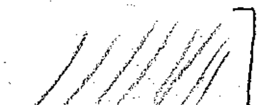
$$\mathcal{T}S = \left\{ f: S \to (M,g) \;\middle|\; \begin{array}{l} f \text{ is a diffeo and } (M,g) \\ \text{has const } -1 \text{ curve.} \end{array} \right\} \Big/ \sim$$

where $f_1 : S \to (M_1, g_1)$ and $f_2 : S \to (M_2, g_2)$ are equivalent if $\exists$ an isometry $h : (M_1, g_1) \to (M_2, g_2)$ s.t.

$$\begin{tikzcd} S \arrow[r, "f_1"] \arrow[dr, "f_2"'] & M_1 \arrow[d, "h"] \arrow[dl, phantom, "\circlearrowleft"] \\ & M_2 \end{tikzcd}$$

commutes up to homotopy, i.e. $h \circ f_1 \simeq f_2$

[Not just the metric but how the surface wears it.]

[Related example: Flat metrics on $T^2$ up to scaling, $\mathcal{T}T^2 =$  ]

[REWORK w/ Metric $d_{g_i}$ as Bill does, this ↓ has issues]

$\mathcal{T}S$ has a topology: $h : (M_1, g_1) \to (M_2, g_2)$ for cpt surf. set:

$$K(h) = \max\left( \sup\left\{ \|Dh(v)\|_{g_2} \;\middle|\; v \text{ a unit vector in } TM_1 \right\}, \sup\left\{ \frac{1}{\|Dh(v)\|_{g_2}} \;\middle|\; v \text{ unit in } TM_1 \right\} \right)$$

$$\Rightarrow \frac{1}{K(h)} d_{g_1}(p,q) \le d_{g_2}(h(p), h(q)) \le K(h)\, d_{g_1}(p,q)$$

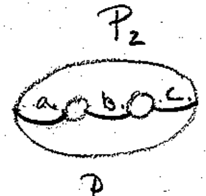
For $f_1, f_2 \in \mathcal{T}S$, set

$$d(f_1, f_2) = \log\left( \inf\left\{ K(h) \;\middle|\; \begin{array}{l} h : M_1 \to M_2 \text{ s.t.} \\ h \circ f_1 \simeq f_2 \end{array} \right\} \right)$$

As $S$ is cpt, this is well def'd.

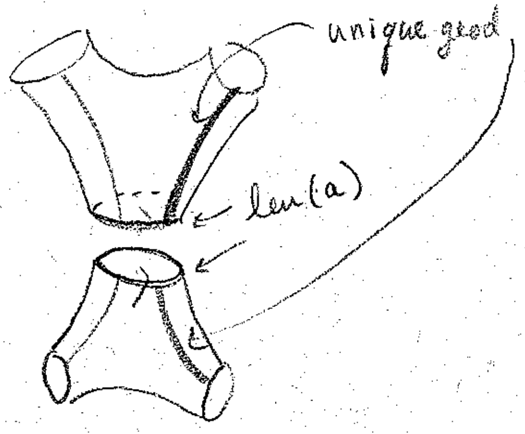
Easy to check that $d$ is a metric ($\triangle$ ineq follows by composition).

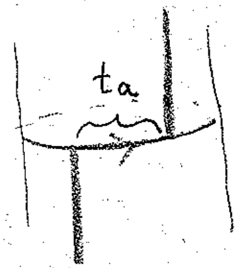
Thm: If $S$ has genus $g$, then $\mathcal{T}S$ is homeo to $\mathbb{R}^{6g-6}$.

Ex: $S =$ 

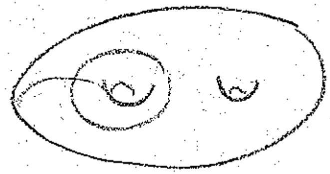
then $\mathcal{T}S$ is 6-dim. Div into pants $P_1$ and $P_2$. By thm, can choose lens of geod to be any $a, b, c \in \mathbb{R}_+$.

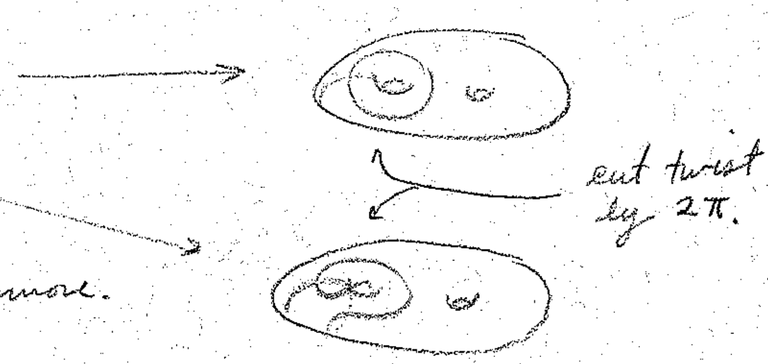
also, have additional "twist" info. Consider first gluing of cuffs:




twist parameter appears to be confined to $[0, a)$. With marking actually varies over $\mathbb{R}$.






orange curves typ. have diff lengths, esp as we twist more and more.

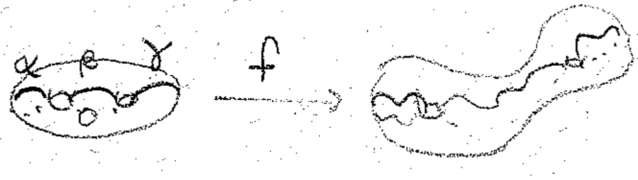
<u>Fenchel-Nielsen Coor:</u>

$$FN: (a, b, c, t_a, t_b, t_c) \in \mathbb{R}^3_+ \times \mathbb{R}^3 \longrightarrow \mathcal{T}S.$$

<u>Claim:</u> FN is a homeo

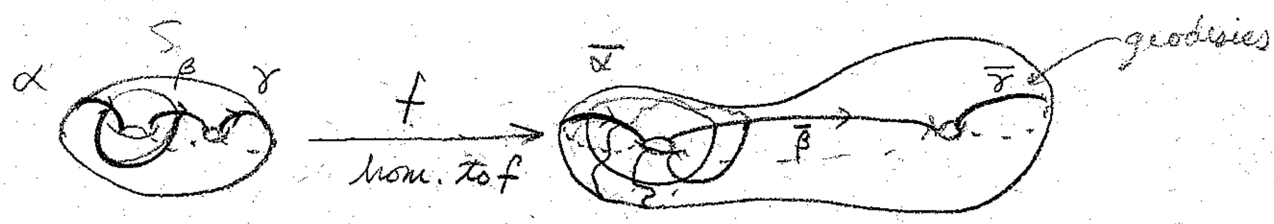
Def $G: \mathcal{T}S \to \mathbb{R}^3_+ \times \mathbb{R}^3$ via:

$f: S \to (M, g)$.

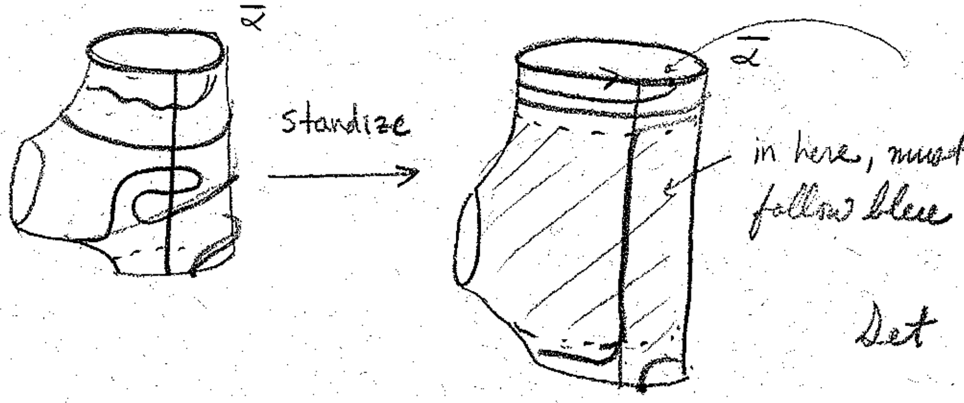



<u>Lemma:</u> Let $\alpha_1, \dots, \alpha_n$ be disjoint circles embedded in a hyp surf $(M, g)$, none of which bound a disk. Then $\exists$ <u>unique</u> geodesics $\gamma_i$ homotopic to $\alpha_i$. Moreover the $\gamma_i$ are disjoint.

[1st bit follows from HW once know that $\alpha_i$ are homot essential, give plain argument for disjointness.]




— = image of marking arc. — uniqueness of geod.




have a real valued winding # for red arc (e.g. 2.1 (len(a)) in pie))

Let $t_{\bar\alpha}$ = diff of winding numbers on two sides of $\bar\alpha$.

Let $G(f) = (\mathrm{len}(\bar\alpha), \mathrm{len}(\bar\beta), \mathrm{len}(\bar\gamma), t_{\bar\alpha}, t_{\bar\beta}, t_{\bar\gamma})$

By uniqueness of geod reps — and of min arcs in pairs of pants, $G$ is well defined. We have

$$G \circ FN = id \quad [\text{pretty much clear}]$$

and
$$FN \circ G = id. \quad [\text{similar}]$$

Thus $FN$ is a bijection. Can check that it is a homeo. ■ [cont is clear].

Another point of view: $\mathcal{T}S$ as space of conformal or Riemann surface or complex manifold structures.

Uniformization Thm: $M$ a clsd surface, $g$ a metric on $M$.
Then $\exists f \in C^\infty(M)$ s.t. $(M, fg)$ has const curve $1, 0,$ or $-1$.
If this curve $\neq 0$, then $f$ is unique.

Thus $\mathcal{T}S$ is important in algebraic geometry; $\mathcal{T}S$ itself has a complex str, making it holom. to an open subset of $\mathbb{C}^{3g-3}$ [holo quad diff etc.]

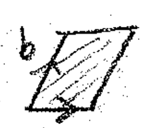
What about $\mathcal{M}S$? $\mathcal{T}S \longrightarrow \mathcal{M}S$ via forgetting the labeling. What is the fiber? $f_1, f_2 : S \longrightarrow (M_1, g_1)$ map to same pt in $\mathcal{M}S$, $f_1 \circ f_2^{-1} \not\simeq \mathrm{id}_S$ if diff pts

Let $\mathcal{MCG}(S) = \mathrm{Diff}(S)/\mathrm{Diff}_0(S) \overset{\text{turns out}}{=} \mathrm{Diff}(S)/\text{homotopy} = \mathrm{Out}(\pi_1 S)$

a countable group. [can argue ↑ is countable directly after observing ~~that observing~~ that the set of hom. classes of s.c.c. is countable]

$\mathcal{MCG}(S)$ acts on $\mathcal{T}S$, and have $\mathcal{M}S$ as the quotient.

Example: Flat tori. $\mathcal{M}T^2$ = flat metrics on $T^2$, up to scale.
$\mathcal{T}T^2$ = marked such.




$f : T^2 \longrightarrow (M, g)$