Math 157a Homework #7 Due Friday, March 9

For these questions, you may want to refer to Section III.L of GHL which is all about hyperbolic geometry. You might also want to consult another source about hyperbolic geometry, such as Thurston's book listed in the course syllabus.

1. Let M be a compact surface with a metric of constant -1 curvature. Prove that there exists a *unique* geodesic in each free homotopy class of loops in M.

2. Let M be a compact surface with $\chi(M) < 0$. If M is non-orientable, prove that M has a metric of constant negative curvature.

3. One of the properties of negative curvature is that you have a *linear isoperimetric inequality*, namely that any closed loop bounds a disk whose area is proportional to the length of the loop. In this problem, you'll show this for the hyperbolic plane $\mathbb{H}^2$.

   To begin, let me clarify what what it means for a loop γ in $\mathbb{H}^2$ to bound a disk if γ is not embedded. Let S^1 be the unit circle in $\mathbb{R}^2$, and D the closed disk that it bounds. If $\gamma\colon S^1 \to \mathbb{H}^2$ is a loop, then a function $g\colon D \to \mathbb{H}^2$ such that $g = \gamma$ on S^1 is a *disk that γ bounds*. The area of such a disk is:

   $$\int_D |g^*(dA)|$$

   where dA is the area form on $\mathbb{H}^2$. The absolute value signs are in the integrand so this is the full area and not some kind of algebraic area.

   Prove that there is a constant C such that for every loop γ in $\mathbb{H}^2$ bounds a disk whose area $\leq C\text{Length}(\gamma)$

4. In a continuation of the last problem, give an example of a complete Riemannian manifold which does not have a linear isoperimetric inequality.