Math 157a Homework #6; Due Friday, March 2

1. Prove Scholium 3.78 from GHL:

   Let M be a complete Riemannian manifold, and p a point in M. Show that $q \in M$ is in the cut-locus of p if and only if at least one of the following holds:

   (a) There exist distinct minimal geodesics joining p to q.

   (b) There is a minimal geodesic joining p to q along which p and q are conjugate.

2. Let M be a complete Riemannian manifold with non-positive sectional curvature. Consider points p and q in M. Prove that there is a unique geodesic in each homotopy class of paths joining p to q.

3. Let M be a complete Riemannian manifold with non-positive sectional curvature. Show that a non-trivial element of $\pi_1(M)$ has infinite order.

4. As in past HWs, say that a Riemannian manifold (M, g) is *algebraically locally symmetric* if $DR = 0$ everywhere. A Riemannian manifold (M, g) is *geometrically locally symmetric* if for each p in M there is a small embedded ball $B_p(\epsilon)$ so that map $\exp(v) \mapsto \exp(-v)$ is an isometry on $B_p(\epsilon)$.

   Prove that these two conditions are equivalent.