

Math 157a Homework #4; Due Friday, February 9

1. Let (M, g) be a Riemannian manifold, and R the curvature tensor. Show that for all tangent vectors a, b, c, d in $T_p M$ we have $R(a, b, c, d) = R(c, d, a, b)$. (This is the omitted part (iii) of Prop. 3.5 of GHL.)
2. Let S^n be the n -sphere with its standard round metric. Let $\text{Isom}(S^n)$ be the group of Riemannian isometries of S^n . Show that $\text{Isom}(S^n) = O(n+1)$.
3. A closed submanifold N of (M, g) is called *totally geodesic* if every minimal geodesic with endpoints in N is contained in N . That is, N is convex in the sense of the last HW.
 - (a) Let N is totally geodesic submanifold of M , and p a point in N . Suppose that P is a plane in $T_p(N)$. Show that $K_N(P) = K_M(P)$, where the former is the sectional curvature measured in N (with the metric inherited from M), and the latter is the sectional curvature measured in M .
 - (b) Show that if N is not totally geodesic then the conclusion of part (a) need not hold.
 - (c) Use part (a) to prove that sectional curvature of S^n is independent of n , as claimed in class.
 - (d) Suppose N is a closed submanifold of M which satisfies the conclusion of part (a) above. Does N have to be totally geodesic?
4. As explained in GHL §2.58, a connection on TM extends to a connection on the space of tensors on M . Thus if R is the curvature tensor of type $(0,4)$, given a vector field X we can talk about its covariant derivative $D_X R$ which is also a $(0,4)$ tensor. If we think of X as one of the inputs, then DR is a $(0,5)$ tensor.

Say that a Riemannian manifold (M, g) is *algebraically locally symmetric* if $DR = 0$ everywhere. A Riemannian manifold (M, g) is *geometrically locally symmetric* if for each p in M there is a small ball $B_p(\epsilon)$ so that map $\exp(v) \mapsto \exp(-v)$ is an isometry on $B_p(\epsilon)$. In later homeworks, you will show that these two conditions are equivalent; the class of Riemannian manifolds satisfying these conditions are called *locally symmetric*. This is one of the most important classes of Riemannian manifolds: It includes manifolds of constant curvature, $\mathbb{C}P^n$, and compact Lie groups with biinvariant metrics. All of these examples just given are locally homogenous, and this is true of locally symmetric spaces in general.

- (a) Let (M, g) be algebraically locally symmetric. Let c be a geodesic in M . Let X, Y, Z be parallel vector fields along c . Prove that $R(X, Y)Z$ is also parallel.
- (b) Suppose M be a connected Riemannian 2-manifold which is algebraically locally symmetric. Prove that M has constant curvature. (The converse is true in any dimension.)