Math 157a Homework #3; Due Friday, February 2

N1 Let (M, g) be a Riemannian manifold. A set S in M is *convex* if every minimal geodesic c with endpoints in S is contained in S.

Now let $p \in M$ and suppose that exp is an embedding on $B_0(r)$ in $T_p(M)$; set $B = \exp(B_0(r))$. Prove or disprove: B must be convex.

N2 Section 2.98 of GHL proves that if (M, g) is a compact Riemannian manifold and $\pi_1(M) \neq 1$ then M has a closed geodesic. Show this is not the case if (M, g) is merely complete. That is, give an example of a complete Riemannian manifold with $\pi_1(M) \neq 1$ which has no closed geodesics. In your example, where does the proof in 2.98 break down?

N3 This problem and the next refer to the notion of *cut locus*, which is described in sections 2.111-114 in GHL.

Let K be a flat Klein bottle. Compute the cut locus of K.

N4 Let (M, g) be a complete Riemannian manifold. Let p be in M. As in GHL §2.111, set

$$I_v = \{t \in \mathbb{R} \mid c_v \text{ is minimal on } [0, t]\}.$$

Consider the function $\rho\colon T_pM \to \mathbb{R}^+ \cup \{\infty\}$ given by $I_v = [0, \rho(v)]$.

a. Prove that ρ is continuous.

b. Suppose M is complete and every v in T_pM has a cut point. Show that M is compact.