Midterm for Math 151b, Winter 2006

Deadline and timelimit: The exam is due at the beginning of class on Friday, February 17. If you won't be in class on Friday, put it in my box outside the main math office prior to 12:45 pm. Beyond this deadline, there is no limit on the amount of time you can spend on this exam.

Disclaimer, Terms, and Conditions: You may not discuss the exam with anyone except myself. You may *only* consult the following:

- The beloved text, Hatcher's *Algebraic Topology*.
- Your class notes.
- Your returned HW sets.

You can use without proof any result in Hatcher through Chapter 3, not including the extra topics §3.A, §3.B, etc.; you may also use the results from §4.1. (If you want to use something from another section, you need to include the proof.) You can also use the result of any HW problem that was assigned, whether or not you did it. You may also use a calculator or computer to ease your labors, and consult non-algebraic-topology math texts (e.g. on algebra or basic analysis or point-set topology) as seems appropriate. While I believe all the questions are stated correctly, there could still be a typo somewhere. Please contact me if you think something is fishy.

Good luck and have fun!

Actual exam: Do all five problems; all questions are weighted equally.

1. Show that the spaces $X = (S^1 \times \mathbb{C}\mathrm{P}^\infty)/(S^1 \times x_0)$ and $Y = S^3 \times \mathbb{C}\mathrm{P}^\infty$ have isomorphic cohomology rings for every choice of coefficients.

2. Let M be a compact connected 3-manifold (without boundary). Consider the decomposition of $H_1(M;\mathbb{Z})$ into $\mathbb{Z}^r \oplus F$ where F is a finite group. Show that $H_2(M;\mathbb{Z})$ is $\mathbb{Z}^r$ if M is orientable and $\mathbb{Z}^{r-1} \oplus \mathbb{Z}/2$ if M is nonorientable. (In particular, if M is nonorientable, it follows that $r > 0$.)

3. A *Lie group* G is an n-manifold with a group law where the group operations are continuous. More exactly, the multiplication map $G \times G \to G$ given by $(g_1, g_2) \mapsto g_1 g_2$ and inverse map $G \to G$ given by $g \mapsto g^{-1}$ are continuous. Examples include $\mathbb{R}^n$ (where the operation is vector addition), $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ (where the operation is multiplication of complex numbers), the n-torus, and matrix groups like $\mathrm{GL}(n,\mathbb{R})$, $SO(n)$, $U(n)$, etc.

   Here's the problem itself: Prove that a Lie group is always orientable (as an n-manifold).

   There is more about Lie groups in sections §3.C and §3.D in Hatcher, but you certainly don't need any of it to do this problem.

4. Given a topological space X, consider the sub-complex $C_b^*(X;\mathbb{R}) \subset C^*(X;\mathbb{R})$ defined by

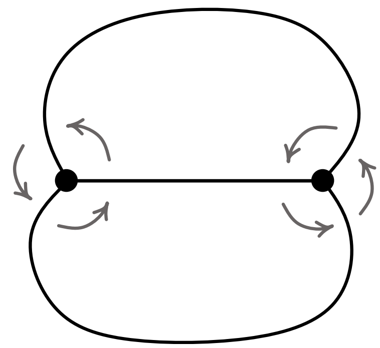
   $$C_b^n(X;\mathbb{R}) = \left\{\phi\colon\ C_n(X) \to \mathbb{R} \;\middle|\; \text{There exists a } K \text{ such that } \left|\phi(\sigma)\right| \leq K \text{ for all } \sigma\colon\ \Delta^n \to X\right\}$$

   The (co)homology of this sub-complex, written $H_b^*(X;\mathbb{R})$, is called the *bounded cohomology* of X with coefficients in $\mathbb{R}$. Bounded cohomology is an example of an exotic cohomology theory, i.e. one which does not satisfy all of the cohomology axioms. Still, it is very useful in certain areas of geometry, e.g. geometric group theory and the study of lattices in Lie groups.

   (a) Describe $H_b^0(X;\mathbb{R})$. How does it differ from $H^0(X;R)$?

(b) Show that $H_b^1(X;\mathbb{R}) = 0$ for any X. Conclude that there is no Mayer-Vietoris sequence for H_b^*, even for finite CW complexes.

(c) Let X be the graph pictured below.



Define a 1-cochain $\phi \in C^1(X;\mathbb{R})$ as follows. Consider the universal cover $\tilde{X}$ of X. If $\sigma\colon [0,1] \to X$ is a 1-simplex, lift σ to $\tilde{\sigma}\colon [0,1] \to \tilde{X}$. As $\tilde{X}$ is a tree, there is a unique (up to reparameterization) embedded path γ in $\tilde{X}$ starting at $\tilde{\sigma}(0)$ and ending at $\tilde{\sigma}(1)$. Define $\phi(\sigma)$ to be the number of times γ "turns right" minus the number of times γ "turns left". (The arrows in the picture show how to make this precise by giving a cyclic order on the edges coming into each vertex.).

Show that while ϕ is not in $C_b^1(X;\mathbb{R})$, its coboundary $\delta\phi$ is in $C_b^2(X;\mathbb{R})$.

(d) Is $[\delta\phi]$ non-trivial in $H_b^2(X;\mathbb{R})$?

5. Hatcher §4.1: #5