Caltech,
Winter 2006

Math 151b: Intro to Alg + Geom Topology!

Jan 4

- Welcome! Pick up syllab, note HW due next Friday.

- Today: Course outline.

151a: [Two inv]

- $\pi_1(X) =$ hom. classes of maps $S^1 \longrightarrow X$

- Homology $H_K(X;G)$

[k dim'l things w/o ∂ / ∂ of k+1 dim'l things]

151b:

Higher homotopy gps: $\pi_n(X) = \overset{\text{hom classes of}}{S^n \longrightarrow X}$

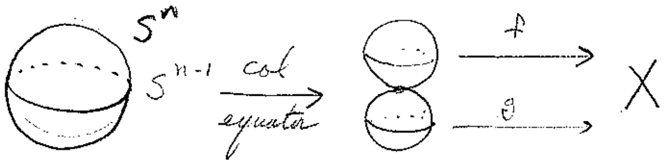
Cohomology: $H^k(X;G)$

[alg dual of homology]

Higher homotopy gps: $\pi_n(X, x_0) =$ hom class of maps $(S^n, s_0) \longrightarrow (X, x_0)$

Group, $[f] * [g] =$

in fact abelian!



n	1	2	3	4	5	6	7
$\pi_n(S^1)$	$\mathbb{Z}$	0	0	0	0	0	0
$\pi_n(S^2)$	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/12$	$\mathbb{Z}/2$

Good: See a great deal of the homotopy type of X.

Whitehead: Suppose $f: X \to Y$ is a map of CW complexes s.t. $f_*: \pi_n(X) \to \pi_n(Y)$ is $\cong$ for all n. Then X and Y are homotopy equivalent.

[Need map here — not enough to just have same π_n]

[Powerful tool]

Bad: Really hard to compute: Don't even know all $\pi_n(S^2)$!

[Query class for methods used in $\pi_1(X)$, H_K. Emph that homology ones all come from excision.]

Cohomology vs. Homology: $X \xrightarrow[f]{} Y$ Top ⟶ Abelian Gps.

$H_k(X;G) \xleftarrow{f^*} H_k(Y;G)$ $H_k(X;G) \xrightarrow{f_*} H_k(Y;G)$

contravariant   covariant

· Defined by a chain complex

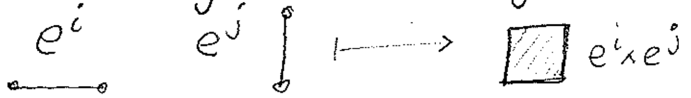
Cohom has similar prop to hom: long exact seq of pair, MV seq

Cohom and Hom determine each other. For a field F, $H_k(X;F) \cong H^k(X;F)$

Key: $H^*(X)$ has a multiplication, – [its algebra, not just an abelian gp]

$H^i(X) \times H^j(X) \longrightarrow H^{i+j}(X)$ [not just is this add'l info, but helps in computations]

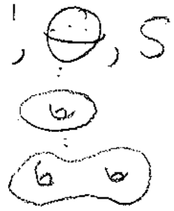
In homology have: $H_i(X) \times H_j(Y) \longrightarrow H_{i+j}(X \times Y)$



Similarly in cohomology $H^i(X) \times \underbrace{H^j(X) \longrightarrow H^{i+j}(X \times X) \xrightarrow{\Delta^*} H^{i+j}(X)}_{\text{cup product}}$

$$\begin{array}{ccc} X & \xrightarrow{\Delta} & X \times X \\ x & & (x,x) \end{array}$$

[one of many applications] ——— ∘ ———

Manifolds: spaces locally homeo to $\mathbb{R}^n$. S^1, , $S^3, S^4, \ldots$

[Query: What did Daniel say about these?]

Smooth mflds: where one can do analysis. $\sum f\, dx_i \wedge dx_j$

Cohomology can be defined in terms of Differential forms.

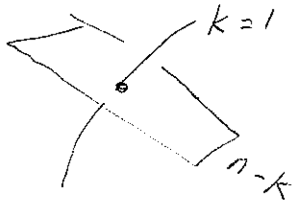
Poincare Duality: Suppose M is a cpt n-mfld. Then $H_k(M;\mathbb{Z}/2) \cong H_{n-k}(M;\mathbb{Z}/2)$.   If M is orient, then $H_k(M;G) \cong H^{n-k}(M;G)$.

The source of P.D. has to do with intersecting submflds.

$$H_k(M;\mathbb{Z}/2)\times H_{n-k}(M;\mathbb{Z}/2)\longrightarrow \mathbb{Z}/2$$ non degenerate

counts $\cap$s mod 2.

$n=3$



May do other stuff, spectral sequences, etc.

Cohomology 101: X space

[Singular/Cellular]

Homology: $\longrightarrow C_{n+1}\xrightarrow{\partial_{n+1}} C_n\xrightarrow{\partial_n} C_{n-1}\longrightarrow$

$C_n = C_n(X;\mathbb{Z}) = \bigoplus_{\sigma_{n}\text{-simplex}} \mathbb{Z}$

Cochains:

$\longleftarrow C^{n+1}\xleftarrow{\delta_n} C^n \xleftarrow{\delta_{n-1}} C^{n-1}\longleftarrow$

$C^n(X;G)=\mathrm{Hom}(C_n,G) = \prod_\sigma G$ = fns from set of simplices to G.

Coboundary: $\delta_n(\varphi)=\varphi\circ\partial_{n+1}$ $\quad\varphi: C_n\to G$

Check: $\delta_n\circ\delta_{n-1}(\varphi) = (\varphi\circ\partial_n)\circ\partial_{n+1}$
$= 0.$

Def: $H^n(X;G) = \ker\delta_n \big/ \mathrm{im}\,\delta_{n-1}$.

Note: $X\xrightarrow{f}Y$ $\quad f^*(\varphi\in C^n(Y;G)) = \varphi\circ f_*$

$C_n(X)\xrightarrow{f_*}C_n(Y)$

$C^n(X)\xleftarrow[f^*]{} C^n(Y)$

Of course, this is a chain map, so induces a map on H^n

Ex: S^1

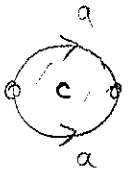
$$C_* \quad 0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0 \qquad C^* \quad 0 \leftarrow \mathbb{Z} \leftarrow \mathbb{Z} \leftarrow 0.$$

3 2 1 0

Ex: $\mathbb{R}P^2$

$$C_* \quad \vdots \quad 0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

$$H_* \quad 0 \cdots 0 \cdots \mathbb{Z}/2 \cdots \mathbb{Z} \cdots 0$$



C^*

$$0 \leftarrow \mathbb{Z} \xleftarrow{\delta_1 = \times 2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \leftarrow 0$$

$\langle \alpha(c)=1 \rangle \quad \langle \varphi : \varphi(a) = 1 \rangle$

$$\delta_1(\varphi)(c) = \varphi \circ \partial_2(c) = \varphi \circ 2a = 2.$$

$$H^* \quad 0 \quad \mathbb{Z}/2 \quad 0 \quad \mathbb{Z} \quad 0$$

Plan for upcoming lectures: **Friday**: examples, way of thinking about cohomology.

Mon: Universal Coeff theorem.

Wed: cup product.

HW #1: 3.1 #1, 6(a), 10, 11.
3.2 #1, 3.

Lecture II: · Reminder: HW #1 on handout.

Last time:

$$\cdots \longrightarrow C_{n+1} \longrightarrow C_n \xrightarrow{\partial} C_{n-1} \longrightarrow \cdots \qquad C_n = C_n(X;\mathbb{Z})$$

$$\cdots \longleftarrow C^{n+1} \longleftarrow C^n \xleftarrow{\delta} C^{n-1} \longleftarrow \cdots \qquad C^n = \mathrm{Hom}(C_n, G)$$

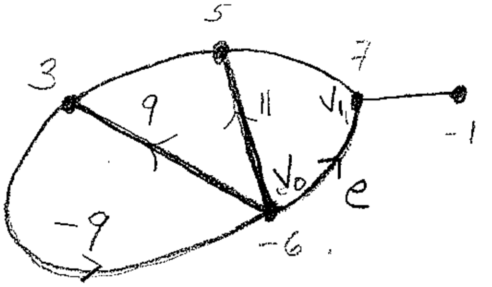
$$\boxed{\delta(\varphi) = \varphi \circ \partial} \qquad H^n(X;G) = {}^{\ker \delta_n}\!/_{\mathrm{im}\,\delta_{n-1}.} \qquad = \{\text{fn from } n\text{-simp} \to G\}$$

Today: Examples / ways of thinking about. · properties.
· Universal Coeff Thm.

* * *

Example: X a graph

$$\begin{tikzcd} 0 & C^1 \arrow[l] \arrow[d, phantom, "\shortparallel"] & C^0 \arrow[l, "\delta"'] \arrow[d, phantom, "\shortparallel"] & 0 \arrow[l] \\ & \text{fn from edges.} & \text{fn from vert to } \mathbb{Z} & \end{tikzcd}$$



$$\delta(\varphi)(e) = \varphi(\partial e) = \varphi(v_1 - v_0)$$
$$= \varphi(v_1) - \varphi(v_0)$$
= difference between values of φ at endpts

[Like the derivative of φ]

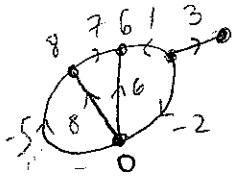
Ker δ = const fns on X^0.

$$H^0(X) = \prod_{\substack{\text{comp} \\ \text{of } X}} \mathbb{Z}$$

Turn to

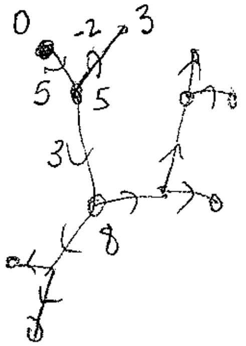
$$H^1 = C^1/_{\mathrm{im}\,\delta}$$

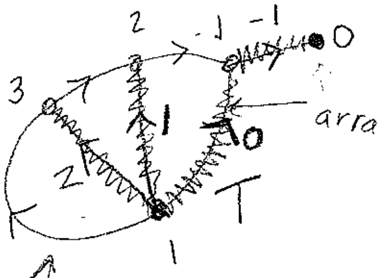
The im δ = those ψ : edges $\to \mathbb{Z}$ can find φ with $\delta\phi = \psi$.



[like finding a fn given its derivative; sol is unique up to adding const fns]

If X is a tree then $H^1 = 0$. In general, pick a maximal tree





$\psi = \delta\varphi$

arrange at will

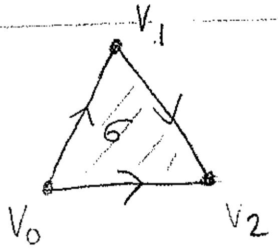
these values fixed by choice here

In particular $C^1/\mathrm{im}\,\delta \cong$ subset of C^1 which vanishes on T

$$\Rightarrow H^1(X) = \prod_{\substack{\text{edges outside} \\ \text{max tree}}} \mathbb{Z} \quad \text{vs. } H_1(X) = \bigoplus_{\text{same}} \mathbb{Z}$$

2-dim'l complex:

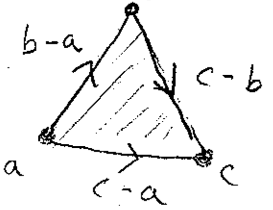
$$C^2 \xleftarrow{\delta} C^1 \ni \psi$$



$$\partial\sigma = [v_0, v_1] + [v_1, v_2] - [v_0, v_2]$$

$$\delta\psi(\sigma) = \psi([v_0, v_1]) + \psi([v_1, v_2]) - \psi([v_0, v_2])$$

= sum of values of ψ around σ. So $\delta\psi = 0 \Leftrightarrow$ $\psi([v_0, v_2]) = \psi([v_0, v_1]) + \psi([v_1, v_2])$



Obstruction: Can we write $\psi = \delta\varphi$ for $\varphi \in C^0$. nee that $\delta\psi = 0$, think of as a local integrability check.

Calc Analogue: vect field, curl, vs. all line integrals, around closed loops. vanish.

Note: $H^1(X;G) = \mathrm{Hom}(\pi_1 X; G)$. } consider omitting.

For basic prop of cohomology: see Hatcher §3.1, part 2.

Reduced cohom: $\tilde{H}^k$ $\qquad C_2 \to C_1 \to C_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0$

$\qquad\qquad\qquad\qquad C^0 \xleftarrow{\varepsilon^*} \mathbb{Z} \leftarrow 0$

Relative cohom: $H^k(X,A)$ with ident def,

long exact seq of a pair

$$\leftarrow H^{n+1}(X,A) \xleftarrow{\delta} H^n(A) \xleftarrow{i^*} H^n(X) \xleftarrow{j^*} H^n(X,A) \leftarrow$$

Excision: $\Big[\ H^n(X,A;G) \to H^n(X-Z, A-Z;G) \quad cl(Z) \subseteq int(A).$

$\Rightarrow \tilde{H}^n(X/A) \cong H^n(X,A)$ for A clsd, nbhd a def. retract. $\Big]$

Simplicial = Cellular, Mayer Vietoris. [All either analogous to homology case, or follow from U.C.T.]
Invariant under homotopy equiv.

Universal Coeff Thm, easy version: Suppose $H_n = H_n(X;\mathbb{Z})$ are finitely gen. Set T_n = torsion subgp of H_n, [i.e. $H_n = \mathbb{Z}^d \oplus T_n$]

$$H^n(X;\mathbb{Z}) \cong H_n/T_n \oplus T_{n-1}.$$

Henceforth $C = \{C_n\}$ be any chain complex of free abelian gps, H_n the homology thereof. Set

$$H^n(C;G) = \text{cohomology of } C^n = \text{Hom}(C;G)$$

Then Thm: $0 \to \text{Ext}(H_{n-1}(C),G) \to H^n(C;G) \xrightarrow{h}$

$$\text{Hom}(H_n(C),G) \to 0.$$

and this splits, but not naturally.

To do: prepare def of h.

$$\begin{tikzcd} & & B_n \arrow[d, phantom, "\cap"] \\ & & Z_n \arrow[d, phantom, "\cap"] \\ {} \arrow[r] & C_{n+1} \arrow[r, "\partial"] & C_n \arrow[r, dashed, "\partial"] & C_{n-1} \arrow[r] & {} \end{tikzcd}$$

$Z_n = \ker \partial$
$B_n = \operatorname{im} \partial$ $\subseteq C_n$

$H_n = Z_n / B_n$

$[\varphi] \in H^n(C;G)$, $\varphi \in \ker \delta \subseteq \mathrm{Hom}(C_n, G)$

Set

$$h([\varphi])([\alpha]) = \varphi(\alpha) \in G.$$

This is well defined as:

1) $\varphi(B_n) = 0$ as $\varphi(\partial b) = \delta\varphi(b) = 0$.

2) if $\varphi = \delta\psi$ then $\varphi(Z_n) = 0$ as
$$\varphi(z) = \delta\psi\, z = \psi(\partial z) = \psi(0) = 0.$$

Claim: h is surjective.

$$\begin{tikzcd} & & \text{free abelian gp} \arrow[d] & \text{free abelian gp} \arrow[d] \\ 0 \arrow[r] & Z_n \arrow[r, shift right, "i"'] \arrow[dr, "id"'] & C_n \arrow[l, shift right, "p"'] \arrow[r, "\partial"] \arrow[d, phantom, "\cong" sloped] & B_{n-1} \arrow[r] & 0 \\ & & Z_n \oplus B_{n-1} \end{tikzcd}$$

$p \circ i = id.$

Should have said why ses splits here.

Given a hom $\varphi_0 \in \mathrm{Hom}(H_n, G)$ get hom $\Rightarrow$

$\varphi_1 : Z_n \to G$ vanishing on B_n $\Rightarrow$

$C^n \ni \psi = \varphi_1 \circ p : C_n \to G$ which is equal to ρ_1 on Z_n

and $\delta\psi = 0$ as φ_1 vanishes on B_n. As $h(\psi) = \varphi_0$ we're done.

Lecture 3

Last time: $C = \{ \to C_{n+1} \to C_n \xrightarrow{\partial} C_{n-1} \to \cdots \}$ a chain complex of free abelian gps.

H_n = homology of C

$H^n(C;G)$ = cohomology of $\leftarrow C^{n+1} \xleftarrow{\delta} C^n \leftarrow C^{n-1} \leftarrow$
$\qquad\qquad\qquad\qquad \mathrm{Hom}(C_n, G)$

Univ Coeff Thm: The following seq is exact

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}, G) \longrightarrow H^n(C,G) \xrightarrow{h} \mathrm{Hom}(H_n, G) \longrightarrow 0$$

$$\underbrace{\varphi \longmapsto (\alpha \longmapsto \varphi(\alpha))}_{\text{is onto}}$$

and moreover splits.

Today: What is Ext and why does it appear here?

Ext: The derived functor of $\mathrm{Hom}(-,G)$ [see if any are familiar with.]

Def: A free resolution of an abelian gp H is an exact seq

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow H \longrightarrow 0$$

where each F_i is free.

Ex: $0 \to \mathbb{Z} \to \mathbb{Z}^2 \to \mathbb{Z} \oplus \mathbb{Z}/2 \to 0$ or
$1 \longmapsto (0,2)$

$\cdots \to \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 0 & 0 \\ 2 & 0 \\ 0 & 2 \end{pmatrix}} \mathbb{Z}^3 \xrightarrow{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}} \mathbb{Z} \oplus \mathbb{Z}/2 \to 0$

Def: $\mathrm{Ext}(H,G)$ is the first cohomology gp of

$$\cdots \longleftarrow F_2^* \longleftarrow \enclose{circle}{F_1} \longleftarrow F_0^* \longleftarrow H^* \longleftarrow 0$$

where $A^* = \mathrm{Hom}(A,G)$ and F_i are any free resolution of H.

Lemma: Let H, H' be abelian gps w/ free resolutions F, F'. Then any hom $\alpha : H \to H'$ extends to a chain map

$$\begin{tikzcd}
\cdots \arrow[r] & F_2 \arrow[r] \arrow[d, dashed] & F_1 \arrow[r] \arrow[d, dashed] & F_0 \arrow[r] \arrow[d, dashed, "\alpha_0"] & H \arrow[r] \arrow[d, "\alpha"] & 0 \\
\cdots \arrow[r] & F_2' \arrow[r] & F_1' \arrow[r] & F_0' \arrow[r] & H' \arrow[r] & 0
\end{tikzcd}$$

and any two such extensions are chain homotopic.

Proof is follow your nose diagram chase.

Chain homotopy

$$\begin{tikzcd}
F_{n+1} \arrow[r] \arrow[d, shift right, "\alpha"'] \arrow[d, shift left, "\beta"] & F_n \arrow[r] \arrow[dl, "h"'] \arrow[d, shift right, "\alpha"'] \arrow[d, shift left, "\beta"] & F_{n-1} \arrow[dl, "h"'] \arrow[d, shift right, "\alpha"'] \arrow[d, shift left, "\beta"] \\
F_{n+1}' \arrow[r] & F_n' \arrow[r] & F_{n-1}'
\end{tikzcd}$$

$$\partial \circ h + h \circ \partial = \alpha - \beta.$$

Note: notationally easier to show that any ind. map for $\alpha = 0$ is chain hom to 0 map

Lemma: For any two free res. F and F' of H, there are canonical homo $H^n(F;G) \cong H^n(F';G)$.

Cor: $\mathrm{Ext}(H,G)$ is well defined.

$\cong H^1(F;G)$

Pf of 2nd lemma:

Consider

$$\begin{tikzcd}
F \arrow[r] \arrow[d, "\alpha"] \arrow[dd, bend right=60, "id"'] & H \arrow[d, "id"] \\
F' \arrow[r] \arrow[d, "\beta"] & H \arrow[d, "id"] \\
F \arrow[r] & H
\end{tikzcd}$$

and

$$\begin{tikzcd}
H^n(F;G) \\
H^n(F';G) \arrow[u, "\alpha^*"] \\
H^n(F;G) \arrow[u, "\beta^*"]
\end{tikzcd}$$

← indep of α.

Thus $\alpha^* \circ \beta^* = 1_{H^n(F;G)}$

$\beta^* \circ \alpha^* = 1_{H^n(F';G)}$ and α^*, β^* are isom, which are canonical by 1st lemma. ▨

Note: Any H has a two step extension

$$0 \to F_1 \to F_0 \to A \to 0$$

$$0 \leftarrow F_1^* \leftarrow F_0^* \leftarrow A^* \leftarrow 0$$

H^* $\qquad \mathrm{Ext}(H,G) \quad 0 \qquad 0$

Turning now to the U.C.T. $0 \to \ker h \to H^n(C;G) \xrightarrow{h} \operatorname{Hom}(H_n(C),G) \to 0$

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
0 \arrow[r] & Z_{n+1} \arrow[r] \arrow[d, "0"] \arrow[dr, phantom, "\circlearrowright"] & C_{n+1} \arrow[r, "\partial"] \arrow[d, "\partial"] \arrow[dr, phantom, "\circlearrowright"] & B_n \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & Z_n \arrow[r] \arrow[d] & C_n \arrow[r, "\partial"] \arrow[d] & B_{n-1} \arrow[r] \arrow[d] & 0 \\
 & {} & {} & {} &
\end{tikzcd}$$

each row is split exact.

Take dual

$$\begin{tikzcd}
 & {} & \varphi_1\circ\partial & \varphi_0 \arrow[l, mapsto] & \\
0 & Z^*_{n+1} \arrow[l] \arrow[u] & C^*_{n+1} \arrow[l] \arrow[dd, bend right] & B^*_n \arrow[l] & 0 \arrow[l] \\
 & & & & \\
0 & Z^*_n \arrow[l] \arrow[uu, "0"] & C^*_n \arrow[l] \arrow[uu, "\delta"] & B^*_{n-1} \arrow[l] \arrow[uu, "0"] & 0 \arrow[l] \\
 & \varphi_0 \arrow[u] & \varphi_1 \arrow[l] \arrow[u] & {} \arrow[u] &
\end{tikzcd}$$

each row is split exact

Take long exact sequence

$$\leftarrow B_n^* \xleftarrow{i_n^*} Z_n^* \leftarrow H^n(C;G) \leftarrow B_{n-1}^* \xleftarrow{i_{n-1}^*} Z_{n-1}^*$$

The boundary map is just i_n^* where $i_n : B_n \to Z_n$ is inclusion.

Hence

$$\begin{tikzcd}
0 & \operatorname{Ker} i_n^* \arrow[l] & H^n(C;G) \arrow[l, "h"'] & \operatorname{coker} i_{n-1}^* \arrow[l] & 0 \arrow[l] \\
 & \boxed{\operatorname{Hom}(H_n;G)} \arrow[u, phantom, "\cong"] & & \operatorname{Ext}(H_{n-1},G). \arrow[u, phantom, "\cong"] &
\end{tikzcd}$$

Point:

$$0 \to B_{n-1} \xrightarrow{i_{n-1}} Z_{n-1} \to H_{n-1} \to 0$$

$$0 \leftarrow B^*_{n-1} \xleftarrow{i^*_{n-1}} Z_{n-1} \leftarrow H_{n-1} \leftarrow 0$$

Splits: $$\begin{tikzcd} 0 \arrow[r] & Z_n \arrow[r, shift right] & C_n \arrow[l, shift right, "p"'] \arrow[r, "\partial"] & B_{n-1} \arrow[r] & 0 \end{tikzcd}$$

gives a map back: $H^n \longrightarrow Hom(H_n, G)$

Rules: $Ext(H \oplus H', G) \cong Ext(H, G) \oplus Ext(H', G)$

$Ext(H, G) \cong 0$ if H is free

$Ext(Z_n, G) \cong G/nG$.

Lecture 4: Today: Cup products and U.C.T for homology.

Cup product:

$$\begin{tikzcd} H^i(X) \times H^j(X) \arrow[r] \arrow[rr, bend right, "\smile"'] & H^{i+j}(X \times X) \arrow[r, "\Delta^*"] & H^{i+j}(X) \end{tikzcd}$$

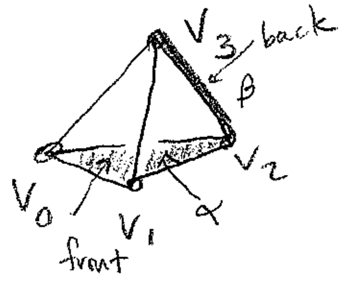
[Intuitively nice but a little messy, see §3.B]

R a ring $\mathbb{Z}$, $\mathbb{Z}/n$, $\mathbb{Q}$, etc. Now define

$$\begin{array}{ccc} C^k(X;R) \times C^l(X;R) & \longrightarrow & C^{k+l}(X;R) \\ \alpha \qquad\qquad \beta & & \alpha \cup \beta \end{array}$$

by

$$(\alpha \cup \beta)(\sigma) = \alpha(\sigma|_{[v_0,\dots,v_k]})\,\beta(\sigma|_{[v_k \ \dots, v_n]})$$



$k=2$, $l=1$

where $\sigma : \Delta^{k+l} \to X$ is in $C^{k+l}(X;R)$.

Lemma: $\delta(\alpha \cup \beta) = \delta\alpha \cup \beta + (-1)^k \alpha \cup \delta\beta$

$\sigma : \Delta^{k+l+1} \to R$

Pf: $(\delta\alpha \cup \beta)(\sigma) = \underbrace{\delta\alpha}_{\alpha \circ \partial}(\sigma|_{[v_0,\dots v_{k+1}]})\,\beta(\sigma|_{[v_{k+1},\dots, v_{k+l+1}]})$

$$\left[\sum_{i=0}^{k+1}(-1)^i \alpha(\sigma_{[v_0,\dots,\hat{v}_i,\dots,v_{k+1}]})\right]\beta(\sigma|_{[v_{k+1},\dots,v_{k+l+1}]})$$

$$(-1)^k \alpha \cup \delta\beta = (-1)^k \alpha(\sigma_{[v_0,\dots,v_k]})\left[\sum_{i=k+1}^{k+l+1}(-1)^{(i-k+1)}\beta(\sigma|_{[v_{k+1},\dots \hat{v}_i,\dots,v_{k+l+1}]})\right]$$

Last term of first cancels w/ first term of second.

Not: • α, β cocycles then so is $\alpha \vee \beta$

• if $\alpha = \delta\varphi$ then

$$\alpha \cup \beta = \delta\varphi \cup \beta = \delta(\varphi \cup \beta)$$

• if $\beta = \delta\varphi$

$$\alpha \cup \beta = \alpha \cup \delta\varphi = (-1)^k \delta(\alpha \cup \beta)$$

$$\Longrightarrow \quad H^k(X;R) \times H^\ell(X;R) \xrightarrow{\cup} H^{k+\ell}(X;R)$$

• associative, distributive; follows from chain level.

Point: Set $H^*(X;R) = \bigoplus_n H^n(X;R)$

This is a ring under cup product. [Note: not nec. comm.]

Moreover, if R has a unit 1_R then so does H^*

$$1_{H^*} \in H^0(X;R) \text{ defined by } 1_{H^*}(0\text{-simplex}) = 1_R$$

Ex: $\mathbb{C}P^2$ CW cell structure

$$0 \to \overset{4}{\mathbb{Z}} \to \overset{3}{0} \to \overset{2}{\mathbb{Z}} \to \overset{1}{0} \to \overset{0}{\mathbb{Z}} \to 0$$

	H^n	gens
4	$\mathbb{Z}$	β
3	0	
2	$\mathbb{Z}$	α
1	0	
0	$\mathbb{Z}$	1

$1 \wedge \alpha = \alpha \qquad \alpha \wedge \beta = \beta \wedge \alpha = 0 \qquad \beta \wedge \beta = 0$

$\boxed{\alpha \wedge \alpha = \beta} \qquad H^*(\mathbb{C}P^2) \cong \mathbb{Z}[\alpha]/\alpha^3$

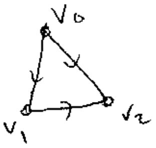
Ex: $S^2 \vee S^4$ same H^*, but $\alpha \wedge \alpha = 0$.

[Hmm... this means its hard to have a cellular def
However there is a simplicial version.]

Ex: $S =$  .

Note: order of vertices important.

[Must respect axiom of Δ complex: order of verts agree w/ that of each sub, i.e.



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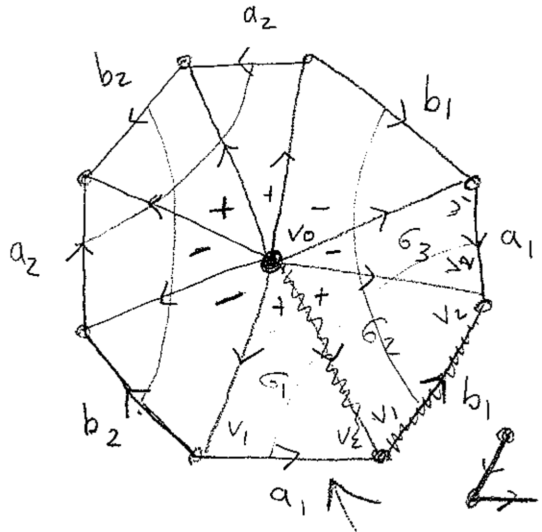
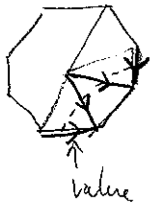


$$H^1(S;\mathbb{Z}) = \mathrm{Hom}(H_1(S;\mathbb{Z}),\mathbb{Z})$$



← rep of α_1, say why δ is 0.

$\alpha_1, \beta_1, \alpha_2, \beta_2$ dual basis to a_1, b_1, a_2, b_2.

$\alpha_1 \wedge \alpha_1(\sigma_1) = \alpha_1(\swarrow)\,\alpha_1(\rightarrow) = 0 \cdot 1 \qquad \alpha_1 \wedge \alpha_1(\sigma_3) = 0$

$\alpha_1 \wedge \alpha_1(\sigma_2) = \alpha_1(\searrow)\,\alpha_1(\nearrow) = 1 \cdot 0 \implies \alpha_1 \wedge \alpha_1 = 0$

a gen [$\alpha_1 \wedge \beta_1(\sigma_2) = 1$

all others 0. $\boxed{\alpha_1 \wedge \beta_1 = \gamma}$

α_1

β_1

$$H^2(S;\mathbb{Z}) = \mathrm{Hom}(H_2(S;\mathbb{Z}),\mathbb{Z}) \cong \mathbb{Z}$$

$\implies \alpha_1 \wedge \beta_1 = \gamma \qquad \underset{[\gamma]}{\cup}$ thus

this is gen

└ given by sign in

$$\beta_1 \wedge \alpha_1(\sigma_3) = 1 \implies \beta_1 \wedge \alpha_1 = -\gamma$$

So

$$\alpha_i \cup \beta_j = \begin{Bmatrix} \gamma & \text{if } i=j \\ 0 & \text{otherwise} \end{Bmatrix} = -\beta_j \wedge \alpha_i$$

Lecture 5:

Univ. Coeff Thm: C a chain complex of free abelian gps then there is a split exact sequence

$$0 \longrightarrow H_n(C)\otimes G \longrightarrow H_n(C;G) \longrightarrow \mathrm{Tor}(H_{n-1}(C),G) \to 0$$

Where: $\mathrm{Tor}(A,B) \cong \mathrm{Tor}(B,A)$

$\mathrm{Tor}(\oplus_i A_i, B) \cong \oplus_i \mathrm{Tor}(A_i, B)$

$\mathrm{Tor}(A,B) = 0$ if A is torsion free

$\mathrm{Tor}(\mathbb{Z}/n, B) \cong \ker(B \xrightarrow{n} B)$.

See §3.A for details.

Suppose $H_n(C) \cong \mathbb{Z}^d \oplus \mathbb{Z}/p^{k_1} \oplus \mathbb{Z}/p^{k_2} \ldots \mathbb{Z}/p^{k_n} \oplus$ other torsion coprime to p.

$H_{n-1}(C) \cong \mathbb{Z}^e \oplus \mathbb{Z}/p^{l_1} \oplus \ldots \oplus \mathbb{Z}/p^{l_m} \oplus$ other torsion

Then $H_n(C;\mathbb{Z}/p) = (\mathbb{Z}/p)^{d+n+m}$.

Ex $\mathbb{R}P^n$

Work one prime at a time

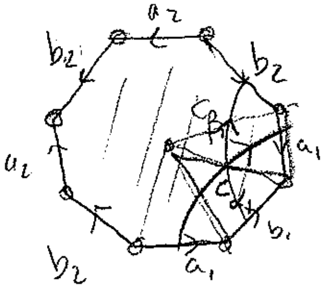
$\mathbb{Z}_2$
0
$\mathbb{Z}_2$
$\mathbb{Z}$

Proof: actually take free resolution

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow A \to 0$$

and $\otimes G$ to get $\mathrm{Tor}(A,G) = H_1$ of.

Last time:



$$\alpha_i \cup \beta_j = \begin{cases} \gamma & i=j \\ 0 & \end{cases} = -\beta_j \cup \alpha_i$$

all others 0.

N.B. equal to intersection # of $c_{\alpha_i} \cap c_{\beta_i}$

$\alpha_i \longleftrightarrow c_{\alpha_i}$

$H^1 \cong H_1$

$\cup$ intersection #

Additional properties:

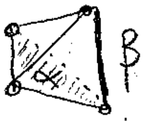
$$\begin{tikzcd} H^k(X,A) \times H^l(X,A) \arrow[r, "\cup"] & H^{k+l}(X,A) \\ H^k(X,A) \times H^l(X) \arrow[ru, "\cup"] & \\ H^k(X) \times H^l(X,A) \arrow[ruu, "\cup"'] & \end{tikzcd}$$

$$\underset{\alpha}{H^k(X)} \times \underset{\beta}{H^l(X,A)}$$

Since

$$C^n(X,A) = \mathrm{Hom}(C_n(X,A) = C_n(X)/C_n(A), R) = \varphi \in C^n(X) \text{ which vanish on } C_n(A).$$

$\alpha \cup \beta(\sigma \in C_n(A))$
$= 0$



Respects induced hom $X \xrightarrow{f} Y$ $\quad H^*(X) \xleftarrow{f^*} \underset{\alpha,\,\beta}{H^*(Y)}$

$$f^*(\alpha \cup \beta) = f^*(\alpha) \cup f^*(\beta)$$

[external cup product]

Cross product: $H^k(X) \times H^l(Y) \xrightarrow{\times} H^{k+l}(X \times Y)$

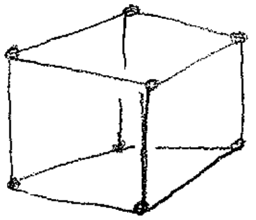
$$\alpha, \beta \longmapsto p_1^*(\alpha) \cup p_2^*(\beta)$$

$$\begin{tikzcd} X \times Y \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \\ X & \end{tikzcd}$$

$$H^k(X,A) \times H^l(Y,B) \xrightarrow{\times} H^{k+l}(X \times Y, A \times Y \cup X \times B)$$

$\underline{\mathcal{E}x}$: $T^n = S^1 \times \cdots \times S^1 = \mathbb{R}^n / \mathbb{Z}^n$.

$$\begin{tikzcd} T^n \arrow[d, "p_i"] \\ S^1 \end{tikzcd}$$ proj onto i^{th} factor



0	1
1	3
2	3
3	1

Fix $\alpha \in H^1(S^1)$ a gen, set $\alpha_i = p_i^*(\alpha)$ then

$H^*(T^n)$ = exterior algebra on $\alpha_1, \ldots, \alpha_n$, $\alpha_i \cup \alpha_j = -\alpha_j \cup \alpha_i$

$H^k(T^n)$ = is the free R-module with basis $\alpha_{i_1} \cup \alpha_{i_2} \cup \cdots \cup \alpha_{i_k}$
$i_1 < \cdots < i_k$.

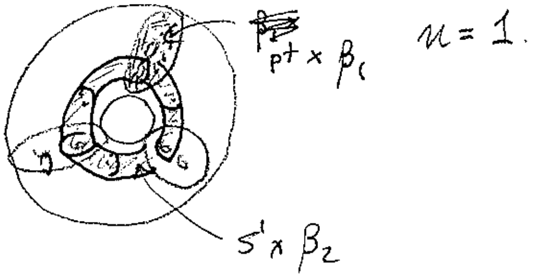
Related to cross product: $n = 4$ $\alpha_1 \cup \alpha_3 = \alpha \times 1 \times \alpha \times 1$
$\alpha_1 \cup \alpha_2 = \alpha \times \alpha \times 1 \times 1$

(*) $H^*(T^n)$ is the free R-module on $\{c_1 \times \cdots \times c_n \mid c_i = 1, \alpha\}$

Lemma: $H^{n+1}(Y) \times H^n(Y) \longrightarrow H^{n+1}(Y \times S^1)$ is an isomorphism

$$(\beta_1, \beta_2) \longmapsto \beta_1 \times 1 + \beta_2 \times \alpha$$

Think about homology:

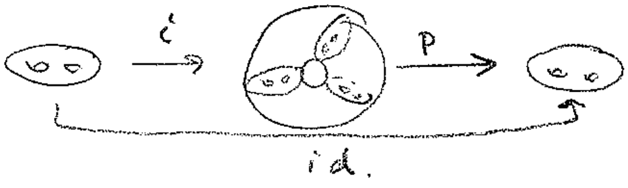


[Point: Every cycle decomposes as the sum of two product type cycles.]

How the lemma applies: prove (*) inductively

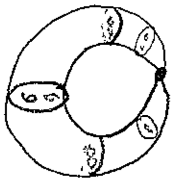
Pf of lemma: Note that
$$\begin{tikzcd} H^{n+1}(Y) \arrow[r, "p^*", "proj"'] & H^{n+1}(Y \times S^1) \arrow[r, "i^*", "inc"'] & H^{n+1}(Y) \\ \beta_1 & \beta_1 \times 1 = p^*(\beta_1) & \end{tikzcd}$$

is an isom as



A little messing w/ long exact seq of pair $(Y\times S^1, Y\times\{pt\})$ shows that it is enough ⑪
to understand

$Y\times S^1/_{Y\times pt}$



$\cong \quad (Y\times I, Y\times\partial I)$

and show $H^n(Y) \to H^{n+1}(Y\times I, Y\times\partial I)$ is an isom

$\beta \longmapsto \beta\times\alpha \qquad \alpha\in H^1(I,\partial I) \cong H^1(S^1).$

$H^0(\partial I) = \langle 1_0, 1_1\rangle$

$\delta|_{\{\beta\times 1_0\}}$ is an isom.

$\beta\times(1_0+1_1)$

$\beta\times 1_0 \quad \beta\times 1_1$

$$\begin{tikzcd}
 & & H^n(Y)\oplus H^n(Y) & H^n(Y) \arrow[l, "\Delta"'] & \\
0 & & & & 0 \arrow[dl] \\
{} & H^{n+1}(Y\times I, Y\times\partial I) \arrow[l] \arrow[ul] & H^n(Y\times\partial I) \arrow[l, "\delta", "\cong"'] & H^n(Y\times I) \arrow[l] & H^n(Y\times I, Y\times\partial I) \arrow[l] \arrow[u] \\
 & H^n(Y)\times H^1(I,\partial I) \arrow[u, "\times"] & H^n(Y)\times H^0(\partial I) \arrow[l, "\mathrm{id}\times\delta"'] \arrow[u, "\times"', "\cong"] & &
\end{tikzcd}$$

$1_0, 1_1$

$$\begin{tikzcd}
 & \mathbb{Z}\oplus\mathbb{Z} & \mathbb{Z} \arrow[l] & 0 \\
H^1(I,\partial I) & H^0(\partial I) \arrow[l] & H^0(I) \arrow[l] & H^0(I,\partial I) \arrow[l] \\
 & & & \overset{?}{\parallel} \\
 & & & \tilde{H}(I/\partial I)
\end{tikzcd}$$

$\longleftarrow 1_0 - 1_1 \longleftarrow$

$\delta(1_0) = (\text{unit})\,\alpha$

$$H^n(Y)\times 1_0 \cong H^{n+1}(Y\times I, Y\times\partial I)$$

Lecture 6: Today: More on cup product.

Thm: $\alpha \in H^k(X)$, $\beta \in H^l(X)$ then $\alpha \cup \beta = (-1)^{kl} \beta \cup \alpha$.

$\alpha \cup \alpha = -\alpha \cup \alpha$

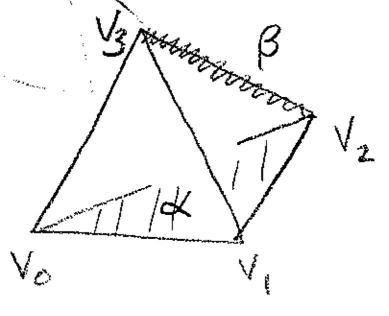
Cor: if k is odd then $2(\alpha \cup \alpha) = 0$ in H^{2k}. If H^{2k} has no 2-torsion then $\alpha \cup \alpha = 0$.

[Cor: If all cohomology is in even degrees, then it is commutative,
or if using $\mathbb{Z}/2$ coeff]

$\sigma : [v_0, v_1, \dots, v_n] \to X$

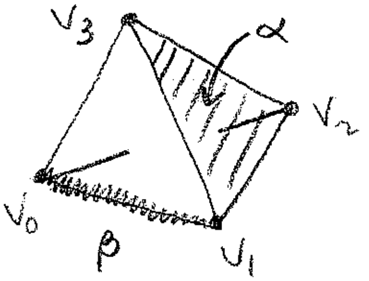
$\alpha \cup \beta(\sigma) = \alpha(\text{front})\beta(\text{back})$

$\beta \cup \alpha(\sigma) = \beta(\text{front})\alpha(\text{back})$



$k=2,\ l=1$

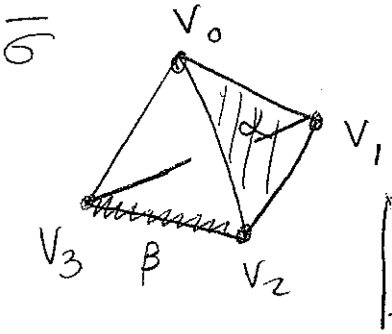
VS.



similar

$\bar{\sigma} = \sigma \circ \left(\begin{array}{l} \text{linear map } \textcircled{τ} \\ [v_0, v_1, v_2, \dots, v_n] \longmapsto [v_n, v_{n-1}, \dots, v_0] \end{array} \right)$

$\bar{\sigma}$



$\alpha \cup \beta(\bar{\sigma}) = \alpha(\overline{\text{back of } \sigma})\,\beta(\overline{\text{front of } \sigma})$

$$\boxed{\varepsilon_{k+l}\, \rho^*(\alpha \cup \beta) = \varepsilon_k \varepsilon_l\, \rho^*(\beta) \cup \rho^*(\alpha)}$$

$\rho : C_n(X) \to C_n(X)$ by $\rho(\sigma) = \varepsilon_n \bar{\sigma}$

$\underbrace{\varepsilon_n = (-1)^{\frac{n(n+1)}{2}}}$ [whether \textcircled{τ} preserves reverses orientation.]

Check: ρ is chain homotopic to id.

Cor ρ^* on C^* is chain hom to id.

As ρ^* induces the trivial map on homology,

hence $x \cup \beta = \varepsilon_{k+l}\,\varepsilon_k\,\varepsilon_l\; \alpha \cup \beta$.

* * *

Ex: $H^*(\mathbb{R}P^n;\mathbb{Z}/2) \cong \mathbb{Z}_2[\alpha]/\alpha^{n+1}$ where α is in degree 1.

1 cell in each dim, ∂, δ are trivial. [mod 2]

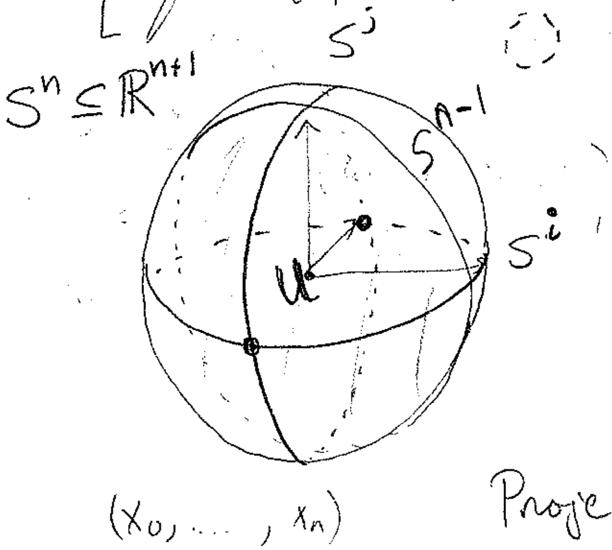
$[H^k(\mathbb{R}P^n, \mathbb{Z}/2) = \mathbb{Z}/2 \quad 0 \leq k \leq n.]$

$[P^n = \mathbb{R}P^n$, homology w/ mod 2 coeff.$]$

Claim α_i gen $H^i(P^n)$ $\quad i+j=n$
α_j gen $H^j(P^n)$
then $\alpha_i \cup \alpha_j$ gen $H^n(P^n)$

Suffices as $P^{n-1} \longrightarrow P^n$ induces a isom on cohomology in dim $< n$.

[by naturality of cup prod, this establishes alg structure except
for $\alpha_{i-1} \cup \alpha_i$]



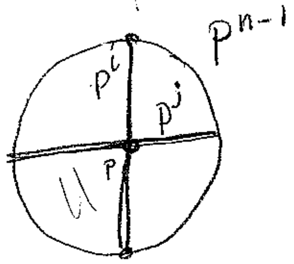
$$S^i = (\overbrace{x_0, \dots, x_i}^{i+1}, \overbrace{0,0,0..}^{j}) \subseteq S^n$$

$$S^j = (\overbrace{0, \dots 0}^{i}, \overbrace{x_i, x_{i+1}, \dots, x_n}^{j+1}) \subseteq S^n$$

$$S^i \cap S^j = S^0 = (0, \dots, 0, \overset{i\text{th place}}{\pm 1}, 0, \dots 0)$$

$$S^{n-1} = (x_0, \dots, x_{i-1}, 0, x_j, x_{j+1}, \dots) \subseteq S^n$$

Project down to P^n $\qquad U = \{x_i > 0\}$



[mumble about Poincaré duality]

$$P^n = \underset{\mathbb{R}^n}{\underset{\cong}{U}} \cup P^{n-1}$$

$$U \cap P^i \cong \mathbb{R}^i$$
$$U \cap P^j \cong \mathbb{R}^j$$

$$\begin{tikzcd}
H^i(P^n) \times H^j(P^n) \arrow[r, "\cup"] \arrow[dr, phantom, "\circlearrowright"] & H^n(P^n) & \\
H^i(P^n, P^n - P^j) \times H^j(P^n, P^n - p^i) \arrow[u, "\circledast"'] \arrow[r, "\cup"] \arrow[d, "\text{inclusion}^*"] \arrow[dr, phantom, "\circlearrowright"] & H^n(P^n, P^n \setminus \{p\}) \arrow[u, "\cong"'] \arrow[d, "\cong \text{ by excision}"] \arrow[r, phantom, "\cong"] & H^n(P^n, P^{n-1}) \\
H^i(\mathbb{R}^n, \mathbb{R}^n \setminus \mathbb{R}^j) \times H^j(\mathbb{R}^n, \mathbb{R}^n \setminus \mathbb{R}^i) \arrow[r, "\cup"] & H^n(\mathbb{R}^n, \mathbb{R}^n - \{0\}) &
\end{tikzcd}$$

as P^{n-1} retracts.

$\mathbb{R}^{n-j}, \mathbb{R}^{n-j} \setminus \{0\}$

S^n

Claim: 1) vertical maps are isomorphisms $\longrightarrow$ for $(*)$ one has $P^n - P^j$ def retracts to P^{i-1}.

2) bottom row has gen $\cup$ gen $=$ gen.

Really same as

$$H^i(I^i, \partial I^i) \times H^j(I^j, \partial I^j) \xrightarrow{\times} H^n(I^n, \partial I^n)$$

$$\begin{tikzcd}
H^i(T^i, \dot{T}^i) \times H^j(T^j, \partial T^j) \arrow[d, "\cong"'] & \\
H^i(T^i) \times H^j(T^j) \arrow[r, "\times"] & H^n(T^n)
\end{tikzcd}$$

$\alpha \times \cdots \times \alpha$ $\qquad$ $\alpha \times \cdots \times \alpha$

Whew!

Lecture 7: What is $H^*(X\times Y)$?

Starting point

$$\underset{\substack{\rotatebox{90}{$\in$}\\ \alpha}}{H^*(X)} \times \underset{\substack{\rotatebox{90}{$\in$}\\ \beta}}{H^*(Y)} \xrightarrow{\times} H^*(X\times Y)$$

[Coeffs R suppressed.]

$$\alpha\times\beta = P_X^*(\alpha)\cup P_Y^*(\beta)$$

[Naively, might hope this is an isomorphism, but..] E.g.

$X = S^1,\ Y=\{pt\},\ R=\mathbb{Z}$ then $(\mathbb{Z}_{(0)}\oplus\mathbb{Z}_{(1)})\oplus\mathbb{Z}_{(0)} \xrightarrow{\times} \mathbb{Z}_{(0)}\oplus\mathbb{Z}_{(1)}$

$X\times Y = S^1$

Note: cross product is bilinear,

$$(\alpha_1+\alpha_2)\times\beta = \alpha_1\times\beta+\alpha_2\times\beta_2$$
$$\alpha\times(\beta_1+\beta_2) = \alpha_1\times\beta_1+\alpha\times\beta_2$$

thus not a homomorphism as

$$\times\big((\alpha_1,\beta_1)+(\alpha_2,\beta_2)\big) = \times\big((\alpha_1+\alpha_2,\ \beta_1+\beta_2)\big) = \boxed{\alpha_1\times\beta_1}+\alpha_1\times\beta_2+\alpha_2\times\beta_1+\boxed{\alpha_2\times\beta_2}$$

$$\times(\alpha_1,\beta_1)+\times(\alpha_2,\beta_2)$$

[This is another reason why this fails]

Solution: replace × with ⊗.

A, B abelian gps.

$$A\otimes B = \left\{\begin{matrix}\text{gp generated by elts}\\ a\otimes b\end{matrix}\right\} \Big/ \begin{matrix}(a+a')\otimes b = a\otimes b + a'\otimes b\\ a\otimes(b+b') = a\otimes b + a\otimes b'\end{matrix}$$

$$\cong \bigoplus_{a,b\in A\times B}\mathbb{Z}[a\otimes b]$$

Ex: $\underset{a_1,\ a_2}{A=\mathbb{Z}\oplus\mathbb{Z}}$ $\quad \underset{b_1,\ b_2\ \ b_3}{B=\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}}$ $\quad A\otimes B\cong\mathbb{Z}^6$ with basis $\{a_i\otimes b_j\}$

Key: $\varphi : A \times B \longrightarrow C$ is a bilinear map we get a homom. $A \otimes B \longrightarrow C$

$$a \otimes b \longmapsto \varphi(a,b)$$

Conversely, such a hom. gives rise to a bilinear map.

Consider

$$H^*(X) \otimes H^*(Y) \xrightarrow{\times} H^*(X \times Y)$$
$$a \otimes b \longmapsto (a \times b)$$

[This is often but not always an isom].

If we set

$$(a \otimes b)(c \otimes d) = (-1)^{|b||c|}(a \cup c) \otimes (b \cup d)$$

then the above map becomes ring homomorphism

Thm: If X and Y are CW complexes and $H^*(Y)$ is a free R-module then $H^*(X) \otimes H^*(Y) \xrightarrow[\times]{} H^*(X \times Y)$ is an isomorphism of rings.

always present when coeff are a field

[Turns out CW cond is unnecessary; freeness is really needed!]

For gen case, see Section 3.B [Involves Tor from U.T.C.]

Did the case when $Y = S^1$.

Ex: $H^*(\mathbb{R}P^n \times \mathbb{R}P^n; \mathbb{Z}/2) = \mathbb{Z}/2[\alpha, \beta]/(\alpha^n, \beta^n)$

[Reason for CW hypothesis is uses axiom char of cohomology.

[Axioms (reduced). $h^n : \mathcal{CW} \to \mathcal{Ab}$ + nat'l coboundary op $\delta : h^n(A) \to h^{n+1}(X/A)$
contravariant functor
sat 1) homotopic maps induce same map on h^n
2) long exact seq of pair 3) $X = \bigvee_\alpha X_\alpha$ $\prod_\alpha h^n(X_\alpha) = h^n(X)$ ← via the obvious isomorphism.]

Thm: If $\mathbb{R}^n$ has the structure of a division algebra, then $n = 2^k$ $[\cong \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}]$

[*Recall:* $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ bilinear. [Need not be comm, fully associative, or have a unit]
and $\forall a \neq 0, b$ we have that $ax = b$ and $xa = b$ are solvable.
$\Longleftrightarrow$ no zero divisors. ← latter for our purposes.

Pf: Set $g : S^{n-1} \times S^{n-1} \to S^{n-1}$ given by $g(x,y) = xy/|xy|$.

$$(-x)y = -(xy) = x(-y) \Longrightarrow g(-x,y) = -g(x,y) = g(x,-y)$$

$\Longrightarrow$ induces a map $h : \underline{P^{n-1}} \times \underline{P^{n-1}} \to \underline{P^{n-1}}$ $[\mathbb{R}P^{n-1} = P^{n-1}]$

What does this do on cohomology?

$$\begin{tikzcd} H^1(P^{n-1} \times P^{n-1}) & H^1(P^{n-1}) \arrow[l, "h^*"'] \\ \alpha + \beta & \gamma \arrow[l, mapsto] \arrow[u, phantom, "\in" rotate=90] \end{tikzcd}$$

where $R = \mathbb{Z}/2$.

γ the gen

where $\alpha = p_1^*(\gamma)$, $\beta = p_2^*(\gamma)$

$$H^1(P^{n-1} \times P^{n-1}) = \underset{\substack{\| \\ \alpha}}{\gamma \otimes 1} \oplus \underset{\substack{\| \\ \beta}}{1 \otimes \gamma}$$

$$H^1(P^{n-1} \times P^{n-1}) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$$

<u>Take $n > 2$</u>, so $\pi_1 P^{n-1} = \mathbb{Z}/2$

$$\begin{tikzcd} P^{n-1} \times P^{n-1} \arrow[r] & P^{n-1} \\ \pi_1 \quad \mathbb{Z}/2 \oplus \mathbb{Z}/2 \arrow[r, "\binom{1}{1}"'] & \mathbb{Z}/2 \end{tikzcd}$$

λ a gen of $\pi_1 P^{n-1}$

$(\lambda, y_0) \longrightarrow \lambda y_0 / |xy|$ ← some linear term

so image is a gen.

Now $\gamma^n = 0$ in $H^*(P^{n-1})$.

$$0 = h^*(\gamma^n) = (\alpha + \beta)^n = \sum \binom{n}{k} \alpha^k \beta^{n-k}$$

Now $H^*(P^{n-1} \times P^{n-1}) \cong \mathbb{Z}/2[\alpha, \beta]/(\alpha^n, \beta^n)$ [as noted above]

so $\binom{n}{k} \equiv 0 \bmod 2$ for all $0 < k < n$.

$\Longrightarrow$ n is a power of 2. ∎

elem
\# theory argument.

In $\mathbb{Z}/2[x]$ have $(1+x)^n = 1 + x^n$ iff $n = 2^l$.

i.e. binary expansion

Pf: $n = n_1 + \cdots + n_k$ when $n_1 < \cdots < n_k$ are all powers of 2,

then $(1+x)^n = (1+x)^{n_1} \cdots (1+x)^{n_k} = \underbrace{(1 + x^{n_1}) \cdots (1 + x^{n_k})}_{\text{has } 2^k \text{ distinct terms.}}$

by binary exansion.

$\Rightarrow (1+x)^n = 1 + x^n \Rightarrow k = 1 \Rightarrow n = 2^l$ ∎

Next: Poincaré Duality

Lecture 8: Homology of Manifolds.

Def: An n-manifold is a Hausdorff, 2nd countable topological space where every pt has an open nbhd homeo to $\mathbb{R}^n$] write up ahead of time.

[Geometric topology: study of such. For now, will be interested in H_* and H^*.]

Poincaré Duality: M a compact, connected n-mfld then $H_k(M;\mathbb{Z}/2) \cong H_{n-k}(M;\mathbb{Z}/2)$

If M is orientable then $H_k(M;\mathbb{Z}) \cong H^{n-k}(M;\mathbb{Z})$

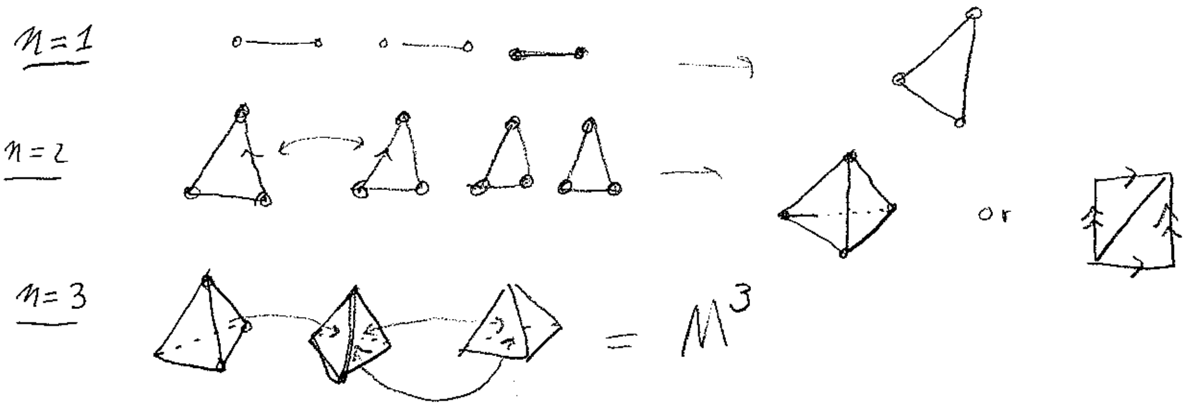
[a little surprising since being a manifold is a purely local cond.]

Today: $H_n(M;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{orientable} \\ 0 & \text{nonorient} \end{cases}$ $\quad H_n(M;\mathbb{Z}/2) = \mathbb{Z}/2.$

[M a cpt connected n-mfld]

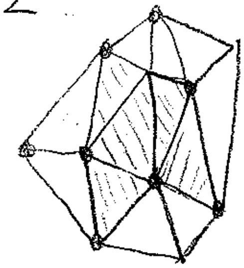
Def: A triangulation of M is a Δ-cplx str consisting of n simplices w/ thier n-1 faces glued in pairs.

Ex:

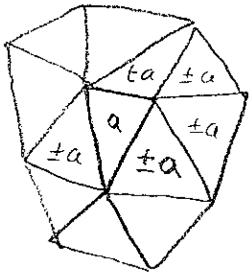


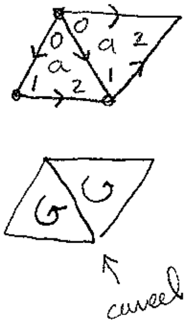
Suppose M has a triangulation: Then $H_n(M;\mathbb{Z}/2) \doteq \mathbb{Z}/2$

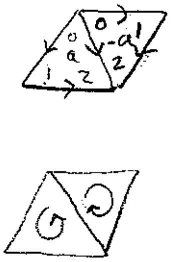
[Q: what is the gen? Q: where do we use conn? cpt?]



What about $H_n(M;\mathbb{Z})$?







$\mathbb{R}P^n$ n odd

maybe everything fits together... $H_n(M;\mathbb{Z}) = \mathbb{Z}$ eg S^2, T^2, 

$H_n(M;\mathbb{Z}) = 0$ e.g $\mathbb{R}P^n$ n even

K^2

$P^2 \# \cdots \# P^2$

and maybe not

← Möbius band.

<u>Q</u>: Does every compact n-mfld have a triangulation?

<u>A</u>: Unknown; however if add geom cond to def of triangulation then the answer is <u>no</u>. [Smooth mflds have triang]

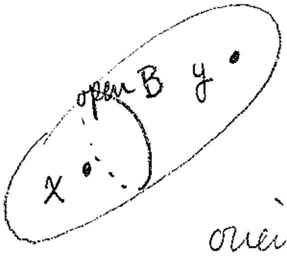
<u>Orientation of $\mathbb{R}^n$</u>: [preserved under rotations but switched under reflections.]

An orientation of $\mathbb{R}^n$ at x is a choice of generator

in $H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \cong H_n(\mathbb{R}^n - \{x\}) \cong H_n(S^{n-1}) = \mathbb{Z}$.

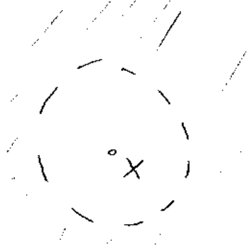
$$\begin{tikzcd}
H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) & \boxed{H_n(\mathbb{R}^n, \mathbb{R}^n \setminus B)} \arrow[l, "\cong"', "i"] \arrow[d, "i"', "\cong \text{ [Query]}"] \\
& H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{y\})
\end{tikzcd}$$

more natural.



orient at one pt determines it at another.

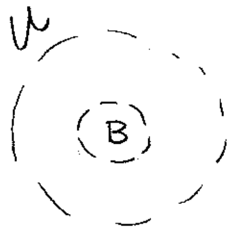
Notation: $H_n(X|A) = H_n(X, X \setminus A)$ "local homology of X at A."



M is an n-mfld. $H_n(M|x) \cong H_n(\mathbb{R}|pt)$ [Q: excision]

A local orientation is a choice of gen $u_x \in H_n(M|x)$.

Def: An orientation of M is a fn $x \mapsto u_x$ s.t.



for every open set $U \cong \mathbb{R}^n$ and bounded open ball B we have $\exists\, u$ s.t

$$\begin{tikzcd}
H_n(M|x) & H_n(M|B) \arrow[l] \arrow[d] & H_n(U|B) \arrow[l, phantom, "\cong"] & \mathbb{Z} \arrow[l, phantom, "\cong"] \\
u_x & u \arrow[l, mapsto] \arrow[d, mapsto] & & \\
H_n(M) \arrow[r] & H_n(Y|x) & & \\
& u_y & &
\end{tikzcd}$$

If such an orientation exists, then M is orientable.

Thm: M clsd conn n-mfld. Then if M is orientable, then $H_n(M;\mathbb{Z}) = \mathbb{Z}$ and $H_n(M;\mathbb{Z}) \longrightarrow H_n(M|x:\mathbb{Z})$ is an isom for all $x \in M$. Otherwise, $H_n(M;\mathbb{Z}) = 0$.

Rmk: Easy to see hom cond $\Rightarrow$ orientability

When M is orient, a gen of $H_n(M;\mathbb{Z})$ is called a fundamental class for M, and denoted $[M]$.

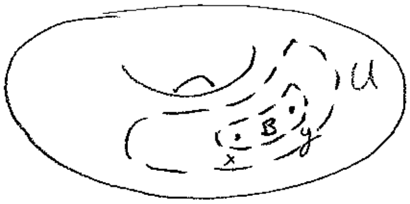
Lecture 9: Last time: $H_n(X|A) = H_n(X, X-A)$ local homology

M an n-mfld. A local orient of M at x is a gen $u_x \in H_n(M|x;\mathbb{Z})$

An orientation of M is a fn $x \mapsto u_x$ sat: $\forall B^{\text{bounded ball}} \subseteq U^{\text{open}} \cong \mathbb{R}^n \subseteq M$ and $x, y \in M$, $\exists\, u$ s.t.

$$\begin{tikzcd} H_n(M|x) & H_n(M|B) \arrow[l] \arrow[d] \\ & H_n(M|y) \end{tikzcd}$$

$$\begin{tikzcd} u_x & u \arrow[l, mapsto] \arrow[d, mapsto] \\ & u_y \end{tikzcd}$$



Today: orientations and $H_n(M;\mathbb{Z})$.

Covering space P.O.V: $\mathbb{R}P^2$

covering map ←

nonorient orientable

sim w/ Klein bottle

↑

torus.

In gen:

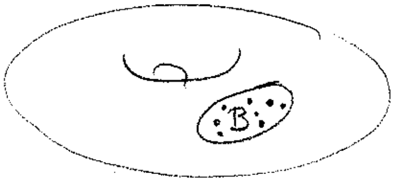
$$\tilde{M} = \{u_x \mid x \in M,\ u_x \text{ a local orient at } x\}$$

Topologize: $B^{\text{finite}} \subseteq \mathbb{R}^{n\ \text{open}} \subseteq M$, u_B a gen of $H_n(M|B)$

Declare to be open:

$$U(u_B) = \{u_x \in \tilde{M} \mid x \in B \text{ and } u_B \mapsto u_x\}.$$

$H_n(M|B) \to H_n(M|x)$



Consider: $p : \tilde{M} \longrightarrow M$
$u_x \longmapsto x$

easy to check that this is a covering map.

$\tilde{M}$ is orientable: $\tilde{u}_x \in H_n(\tilde{M}|u_x) \cong H_n(U(u_B)|u_x) \cong H_n(B|x) \ni u_x$

$u_x \longmapsto \tilde{u}_x$

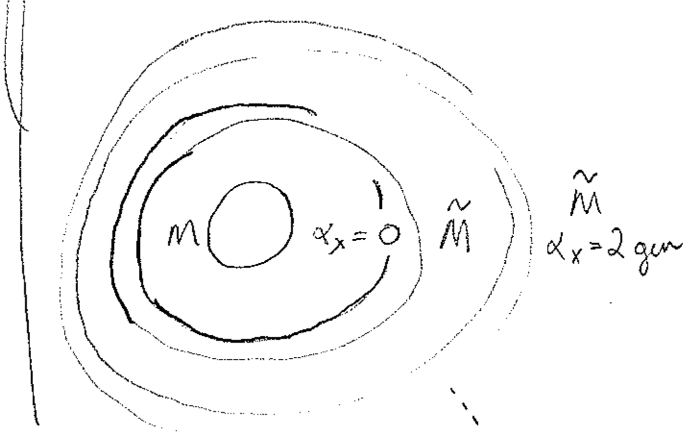
<u>Prop</u>: Suppose M is connected. Then M is orient iff $\tilde{M}$ has two components [talk through, but don't write]

<u>Cor</u>: if $\pi_1 M = 1$, M is orient.

— ∘ —

omit

$M_{\mathbb{Z}} \to M$ where $M_{\mathbb{Z}} = \{\alpha_x \in H_n(M|x)\}$ [top as before]



$$\begin{tikzcd} M_{\mathbb{Z}} \arrow[d, "p"] \\ M \arrow[u, bend left, "s"] \end{tikzcd}$$

is a section if $p \circ s = \mathrm{id}_M$

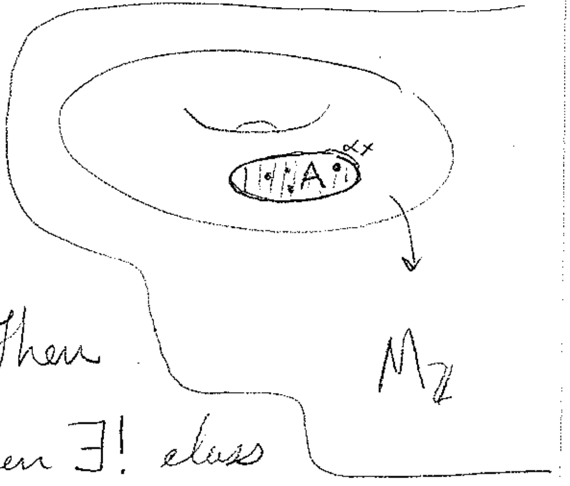
$s(x) = \alpha_x \in H_n(M|x)$

[an orient is a section where each α_x is a gen]

<u>Thm</u>: M closed conn n-mfld. Then M is orient $\Leftrightarrow$ $H_n(M;\mathbb{Z}) = \mathbb{Z}$ and $H_n(M) \to H_n(M|x)$ is an isomorphism $\forall x$.

& if M is nonorient, then $H_n(M) = 0$.

For any M, $H_k(M) = 0 \quad \forall\, k > n$.



<u>Lemma</u>: M an n-mfld, $A^{cpt} \subseteq M$. Then

(a) If $x \mapsto \alpha_x$ is a section of $M_{\mathbb{Z}} \to M$, then $\exists!$ class $\alpha_A \in H_n(M|A)$ whose image in $H_n(M|x)$ is α_x for all $x \in A$.

(b) $H_k(M|A) = 0$ for $k > n$.

Pf of Lemma from Thm: Take $M = A$, orient M. $x \longmapsto u_x$

$\exists!\ \alpha \in \underbrace{H_n(M) \text{ whose image in } H_n(M|x)}_{\text{an isom.}} = u_x \ \forall x.$

Argh! ↑ more complicated

Pf of Lemma:

Step 1: $\boxed{\text{true for } A, B, A \cap B \Rightarrow \text{true } A \cup B.}$ Key

Step 2: Suffices to consider $M = \mathbb{R}^n$

Pf: $A = \bigcup_{i=1}^{n} A_i$, each A_i some chart $U_i \cong \mathbb{R}^n$; For each A_i use excision to see $H_n(M|A_i) \cong H_n(U_i|A_i)$.

Step 3: Holds for convex sets $A \subseteq \mathbb{R}^n$, hence unions thereof as $\cap$ of convex is convex

Pf: Note that $\mathbb{R}^n \setminus x$ and $\mathbb{R}^n \setminus A$ both def retract to a sphere about x.

Thus $H_n(\mathbb{R}^n|A) \xrightarrow[\cong]{} H_n(\mathbb{R}^n|x)$, and all higher H_k vanish.

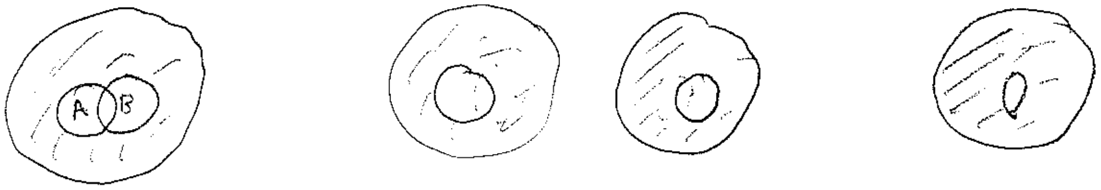
Step 4: Step 3 implies true for any subset of $\mathbb{R}^n$.

Step 1: M-V sequence:

$$\alpha \oplus \beta \longmapsto \alpha + \beta$$

$$\begin{tikzcd}
H_{n+1}(M|A\cap B) \arrow[r] \arrow[d, phantom, "\shortparallel"] & H_n(M|A\cup B) \arrow[r, "\Phi"] & H_n(M|A)\oplus H_n(M|B) \arrow[r, "\Psi"] & H_n(M|A\cap B) \arrow[r] & {} \\
0 & & & &
\end{tikzcd}$$

$$\alpha \longmapsto (i(\alpha), -i(\alpha))$$



All $H_k(M|A\cup B) = 0$ ~~by~~ ~~b~~ for $k > n$. $\Rightarrow$ (b)

$$\begin{tikzcd}
 & & & H_n(M|x) \ni u_x \\
0 \arrow[r] & H_n(M|A\cup B) \arrow[r] & H_n(M|A)\oplus H_n(M|B) \arrow[r] \arrow[urr, dotted] & H_n(M|A\cap B) \arrow[u] \arrow[r] & {} \\
 & & u_A \qquad u_B \arrow[r, bend right] & u_{A\cap B} &
\end{tikzcd}$$

both go here by uniqueness.

given by induction

Set
$$u_{A\cup B} \longrightarrow (u_A, -u_B) \longrightarrow 0$$

Then $u_{A\cup B} \longrightarrow u_x \quad \forall x \in A\cup B$.

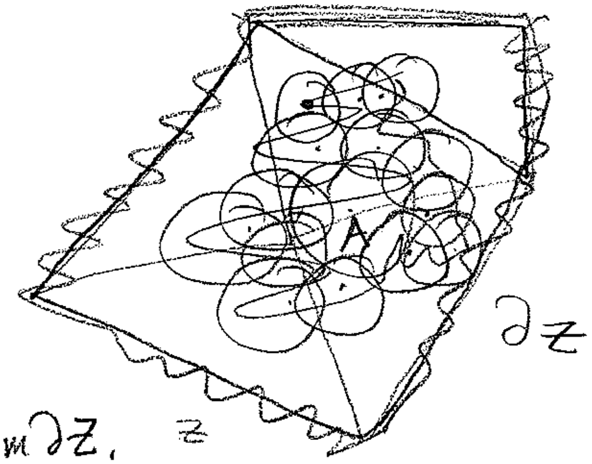
If more than one such, image in $H_n(M|A) \oplus H_n(M|B)$

have def prop. of u_A, u_B, hence equal to it. As Φ is inj, $u_{A\cup B}$ ~~exist~~ unique. ∎

Step 4. $\alpha \in H_k(\mathbb{R}^n|A)$ rep by z.

∂z — cpt disjoint from A,

$d(\partial z, A) = \delta > 0$.



K = Union of a finite # balls of diam δ covering A and disjoint from ∂z.

z is also a rel cycle for $H_k(\mathbb{R}^n/K)$

u_K

$$\begin{tikzcd} & H_k(\mathbb{R}^n/K) \arrow[d] \\ H_k(\mathbb{R}^n|x) & H_k(\mathbb{R}^n|A) \arrow[l] \end{tikzcd}$$

If $k > n$, then $[z] = 0$ in hence in) by 3.

If $k = n$, consider u_K, and set $u_A = i_*(u_K)$; it has the desired prop. Uniqueness follows as if u_A & u_A' are two such elts, consider $[u_A - u_{A'}] = [z]$

Then

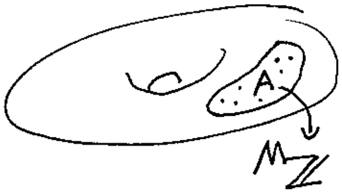
$$\begin{tikzcd} M_{\mathbb{Z}} = \{ \alpha_x \in H_n(M|x;\mathbb{Z}) \} \arrow[d] & \arrow[d, mapsto] \\ M & x \end{tikzcd}$$

Last time:

Lemma: M an n-manifold, $A^{cpt} \subseteq M$. Then

ⓐ If $x \mapsto \alpha_x$ is a section of $M_{\mathbb{Z}} \to M$, then $\exists$ a unique $\alpha_A \in H_n(M|A)$ whose image in $H_n(M|x)$ is α_x for all $x \in A$

ⓑ $H_k(M|A) = 0$ for $k > n$.



Pf: Step 1: [Key] true for $A, B, A \cap B \Rightarrow$ true for $A \cup B$

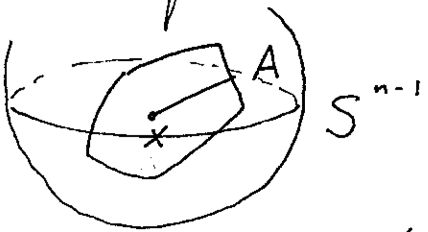
Step 2: Suffices to consider $M = \mathbb{R}^n$.

Pf: $A = \bigcup_{i=1}^{n} A_i$ for each A_i cpt and $\subseteq$ chart U_i

For each A_i use excision to see $H_n(M|A_i) \cong H_n(U_i|A_i)$

Now apply 1.

Step 3: Holds for $A \subseteq \mathbb{R}^n$ a union of convex sets



Pf: Suppose A is convex.

Note that both $\mathbb{R}^n \setminus \{x\}$ and $\mathbb{R}^n \setminus A$ def retract to sphere about x. Thus $H_k(\mathbb{R}^n|A) \cong \begin{cases} \mathbb{Z} & k = 0, n \\ 0 & \text{otherwise} \end{cases}$ proving (b). Moreover $H_n(\mathbb{R}^n|A) \to H_n(\mathbb{R}^n|x)$ is an isom for all $x \in A$, proving (a). [especially the uniqueness.]

Now note $\cap$ of convex is convex, and apply Step 1.

Step 4: Step 3 $\Rightarrow$ lemma for every $\overset{\text{cpt}}{\wedge}$ subset of $\mathbb{R}^n$.

Pf of Step 1: Meyer-V sequence

$$\begin{tikzcd}
 & & \beta \oplus \gamma \arrow[rr, mapsto] & & i(\beta)+i(\gamma) \\
H_{n+1}(M|A\cap B) \arrow[r] \arrow[d, phantom, "\parallel"] & H_n(M|A\cup B) \arrow[r, "\Phi"] \arrow[d, phantom, "\ni"] & H_n(M|A)\oplus H_n(M|B) \arrow[rr, "\Psi"] & & H_n(M|A\cap B) \arrow[r] & {} \\
0 & \alpha \arrow[r, mapsto] & (i_*(\alpha), -i_*(\alpha)) & & &
\end{tikzcd}$$

Thus all $H_k(M|A\cup B)=0$ for all $k>n \Rightarrow$ (b)

For (a) consider $\alpha_A, \alpha_B, \alpha_{A\cap B}$

$$\begin{tikzcd}
 & & & H_n(M|x) \ni \alpha_x & \\
0 \arrow[r] & H_n(M|A\cup B) \arrow[r, "\Phi"] \arrow[urr, bend left] \arrow[d, phantom, "\ni"] & H_n(M|A)\oplus H_n(M|B) \arrow[r] \arrow[ur] & H_n(M|A\cup B) \arrow[r] \arrow[u] & {} \\
 & {} & \alpha_A \quad \alpha_B \arrow[r, bend right] & \alpha_{A\cap B} \arrow[uu, bend right] &
\end{tikzcd}$$

Set $\alpha_{A\cup B}$ to be

the unique elt s.t. $\Phi(\alpha_{A\cup R}) = (\alpha_A, -\alpha_B)$ by uniqueness part of (b).

Then $\alpha_{A\cup B} \rightsquigarrow H_n(M|x)$ for all $x \in A\cup B$

$$\begin{tikzcd}
\alpha_{A\cup B} \arrow[r, rightsquigarrow] \arrow[dr] & H_n(M|x) \arrow[d, phantom, "\ni"] \\
 & \alpha_x
\end{tikzcd}$$

Uniqueness: suppose $\alpha'_{A\cup B}$ also $\mapsto \alpha_x$ $\forall x \in A\cup B$.

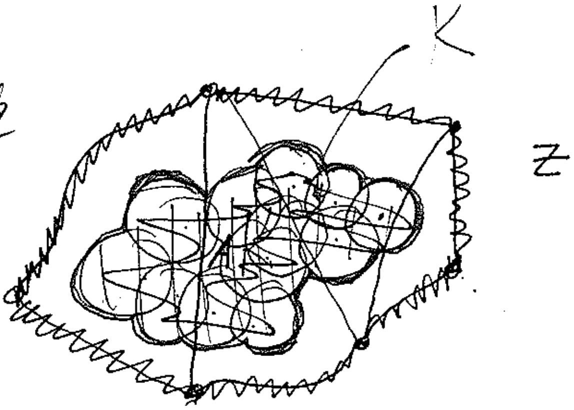
Then $i_*(\alpha'_{A\cup B}) \in H_n(M|A)$ maps to α_x $\forall x \in A$

by uniqueness ~~properties~~ prop of α_A, must have

$i_*(\alpha'_{A\cup B}) = \alpha_A$. Same for B, so $\alpha_{A\cup B} = \alpha'_{A\cup B}$

by inject of Φ.

Step 4: $A \overset{cpt}{\subseteq} \mathbb{R}^n$



$\beta \in H_k(\mathbb{R}^n|A)$ rep by a cycle z. ∂z is cpt, disjoint from A.

$d(\partial z, A) = \delta$. $\Rightarrow$ can cover A with a finite # of closed balls of diam δ, which are disjoint from ∂z. Call the union K.

z is also a rel cycle for

$$\begin{tikzcd}
H_k(\mathbb{R}^n|K) \arrow[d, "i_*"] & {[z] = \beta'} \arrow[d, mapsto] \\
H_k(\mathbb{R}^n|A) & \ni \beta
\end{tikzcd}$$

To see (b) note that

If $k > n$, then

$\beta' = 0$ by step 3 $\Rightarrow \beta = 0$.

Hence $H_k(\mathbb{R}^n|A) = 0$ for $k > n$.

For (b) take $k = n$. $\exists\, \alpha_K$ by Step 3, set $\alpha_A = i_*(\alpha_K)$

this has the right prop w.r.t. points.

to see uniqueness, suppose α_A, α'_A are two such,

consider $\alpha_A - \alpha'_A = [z]$ chose K suitable for z.

Then
$$\begin{tikzcd} H_n(\mathbb{R}^n|K) \arrow[r] & H_n(\mathbb{R}^n|x) \\ {[z]} \arrow[r] & 0 \end{tikzcd}$$
for all x in $\underline{K}$ not A.

(True forall $x \in A$, but for $x, y \in$ Ball get isom)

Thus $[z] = \alpha_{K,\, 0\text{-section}} \overset{\text{by uniqueness}}{\Rightarrow} [z] = 0 \Rightarrow i_*[z] = 0$

$\alpha_A - \alpha_A' = 0 \Rightarrow \alpha_A = \alpha_A'$. ∎

Lecture 11: Poincare Duality 101.

M = cpt conn n-mfld

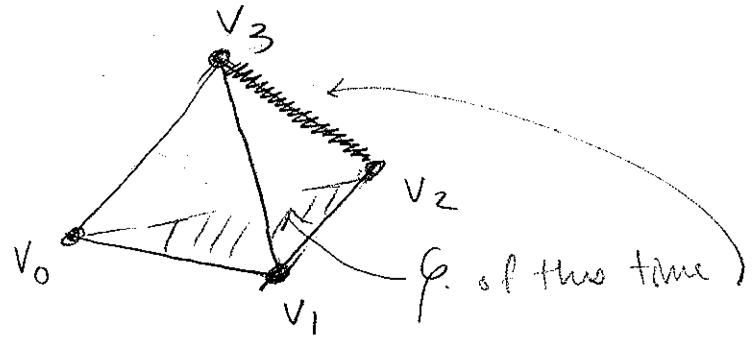
Basic form: If M is orientable, $H_k(M;\mathbb{Z}) \cong H^{n-k}(M;\mathbb{Z})$

[Today: how this relation is mediated, some consequences and interpretations]

Cap product: $\cap : C_k(X) \times C^\ell(X) \longrightarrow C_{k-\ell}(X)$ $\boxed{k \geq \ell}$

$$\underset{\sigma : \Delta^k \to X}{\Psi} \qquad \underset{\varphi}{\Psi}$$

$$\sigma \cap \varphi = \varphi(\sigma|_{[v_0,\dots,v_\ell]})\, \sigma|_{[v_\ell,\dots,v_k]}$$



$k = 3$, $\ell = 2$

Now $\alpha \in C_k(X)$

~~$\partial(\sigma\cap\alpha) = (-1)^\ell(\partial\sigma\cap\alpha - \sigma\cap\delta\alpha)$~~

$\partial(\alpha\cap\phi) = (-1)^\ell(\partial\alpha\cap\phi - \alpha\cap\delta\varphi)$

Note that if σ is a cycle and α a cocycle then $\sigma\cap\alpha$ is a cocycle.

Hence $H_k(X) \times H^\ell(X) \longrightarrow H_{k-\ell}(X)$

Naturality: $X \xrightarrow{f} Y$ $\alpha \in H_k(X)$ $\varphi \in H_\ell(Y)$

$$f_*(\alpha) \cap \varphi = f_*(\alpha \cap f^*(\varphi))$$

Poincare Duality: M orientable $[M] \in H_n(M)$ a gen.

Then $D: H^k(M) \longrightarrow H_{n-k}(M)$ given by

$$D(\alpha) = [M] \cap \alpha \text{ is an isomorphism.}$$

Note: holds true w/ coeffs in R if M is R orientable, in which case $H_n(M;R) = R$

As any mfld is $\mathbb{Z}/2$ orientable, we have

$$\begin{array}{l} H^k(M;\mathbb{Z}/2) \cong H_{n-k}(M;\mathbb{Z}/2) \\ \quad \cong \\ \quad H_k(M;\mathbb{Z}/2) \end{array} \quad (*) \Longrightarrow \text{see next page}$$

Cor: The Euler char of any odd dim'l mfld is 0.

Pf: $$\chi(M) = \sum (-1)^i \operatorname{rank}(H_i(M:\mathbb{Z})) \underset{\uparrow}{=} \sum (-1)^i \dim(H_i(M;\mathbb{Z}/2))$$

2nd eq clear if have a CW decomp Query?

in general, note: $\dim(H^i(M;\mathbb{Z}/2) \cong H_i(M;\mathbb{Z}/2))$

$$= \operatorname{rank}(H_i(M;\mathbb{Z})) + \operatorname{rank}(2\text{-part}) + \operatorname{rank}(2\text{-part in dim } i-1)$$

Using that these are finitely gen

As n is odd, i and $n-i$ have opp. parity.

Hence ______ cancel in pairs.

(*) F field $H^k(X;F) \cong H_k(X;F)^*$, $H_k(X;F) \cong H^k(X;F)^*$

$$C^k(X;F) = \mathrm{Hom}(C_k(X;\mathbb{Z}), F) = \mathrm{Hom}_F(C_k(X;F), F) = C_k(X,F)^*$$

$$H^k(X;F) \longrightarrow (H_k(X;F))^{\overset{\text{for same reason as before.}}{*}} \longrightarrow 0$$

$$H_k(X;F) \longrightarrow (H^k(X;F))^* \longrightarrow 0 \qquad \boxed{V^{**} \cong V}$$

Cup product:

X space

(★) $\psi(\alpha\cap\varphi) \overset{\text{as elts of } R}{\overset{?}{=}} (\varphi\cup\psi)(\alpha)$ $\quad \alpha \in C_{k+\ell}(X;R)$

Since if $\sigma : \Delta^{k+\ell} \longrightarrow X$ $\quad \varphi = C^k(X;R)$

$\psi = C^\ell(X;R)$

$$\psi(\sigma\cap\varphi) = \psi(\varphi(\sigma|_{[v_0,\dots,v_k]})\sigma|_{[v_k,\dots,v_{k+\ell}]}))$$

$$= \varphi(\sigma|_{[v_0,\dots,v_k]})\,\psi(\sigma_{[v_k,\dots,v_{k+\ell}]})$$

$$\varphi\cup\cdot : C^\ell(X) \to C^{k+\ell}(X) \qquad \cdot\cap\varphi : C_{k+\ell}(X) \longrightarrow C_\ell(X)$$

$$\begin{tikzcd}
\psi \quad H^\ell(X,R) \arrow[r, "h"] \arrow[d, "\varphi\cup\cdot"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Hom}_R(H_\ell(X;R), R) \arrow[d, "(\cdot\cap\varphi)^*"] \\
H^{k+\ell}(X,R) \arrow[r, "h"] & \mathrm{Hom}_R(H_{k+\ell}(X;R), R)
\end{tikzcd}$$

$$\mathrm{Hom}_R(C_{k+\ell}(X), R) \xleftarrow{(\cdot\cap\varphi)^*} \mathrm{Hom}_R(C_\ell(X), R) \ni \psi$$

$$(\cdot\cap\varphi)^*(\psi)(\alpha) = \psi(\alpha\cap\varphi)$$

SKIP

$$(\cdot\cap\varphi)^*\circ h(\psi)\,(\alpha\in H_{k+\ell}(X;R)) = \psi((\cdot\cap\varphi)(\alpha)) = \psi(\alpha\cap\varphi)$$

$$((\varphi\cap\cdot)(\psi))(\alpha) = (\varphi\cap\psi)(\alpha).$$

✓

Suppose M is R-orientable, fix $[M] \in H_n(M;R)$

$$\begin{array}{ccccc} H^k(M;R) & \times & H^{n-k}(M;R) & \longrightarrow & R \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \\ \varphi & & \psi & \longmapsto & (\varphi \cup \psi)([M]) \end{array}$$

Such a bilinear map is <u>nonsingular</u> if
$A \times B \xrightarrow{\langle , \rangle} R$

$$\begin{array}{l} a \longmapsto (b \mapsto \langle a, b\rangle) \\ A \longrightarrow \mathrm{Hom}(B,R) \\ B \longrightarrow \mathrm{Hom}(A,R) \end{array}$$

are isomorphisms

<u>Prop:</u> If R is a field, this cup product pairing is nonsingular for each k. If $R = \mathbb{Z}$, it is nonsing if we use $H^*_{\text{free}} = H^*/_{\text{torsion}}$.

<u>Pf:</u>

$$\begin{tikzcd} H^{n-k}(M;R) \arrow[r, "h"] \arrow[rr, bend left] & \mathrm{Hom}_R(H_{n-k}(M;R), R) \arrow[r, "D^*"] & \mathrm{Hom}_R(H^k(M;R), R) \\ \rotatebox{90}{$\in$} & & \\ \underset{\text{fixed}}{\underline{\underline{\psi}}} \arrow[r, mapsto] & h(\psi) \arrow[r, dashed] & D^*(h(\psi))(\varphi) \end{tikzcd}$$

$$\begin{aligned} &= h(\psi)\, D(\varphi) \\ &= \psi([M] \cap \varphi) \\ &= (\varphi \cup \psi)[M] \end{aligned}$$

$$\psi \longmapsto (\varphi \mapsto (\varphi \cup \psi)[M])$$

$$\begin{tikzcd} H^{n-k}(M;R) \arrow[rr] \arrow[dr, "h", "\cong"'] & & \mathrm{Hom}_R(H^k(M;R), R) \\ & \mathrm{Hom}_R(H_{n-k}(M;R), R) \arrow[ur, "D^*"', "\cong"] \arrow[u, phantom, "\curvearrowright"] & \end{tikzcd}$$

Ex: $H^*(\mathbb{C}P^n;\mathbb{Z}) = \mathbb{Z}[\alpha]/(\alpha^{n+1})$ $\quad |\alpha| = 2.$

$\mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^n$ gives an isom on H^* through dim $2n-2$.

Check: $\alpha^n = \alpha \cup \alpha^{n-1}$ is a gen of $H^{2n}(\mathbb{C}P^n;\mathbb{Z})$

$$H^2(\mathbb{C}P^n;\mathbb{Z}) \xrightarrow{F} \mathrm{Hom}_{\mathbb{Z}}(H^{n-2}(\mathbb{C}P^n;\mathbb{Z}),\mathbb{Z}) = \mathbb{Z} \qquad \alpha^{n-1} \longmapsto 1$$

$$\varphi \longmapsto (\varphi \cup \circ)[M] \quad \text{is an isom}$$

$\Rightarrow \exists\, \varphi = m\alpha$ s.t. $F(\varphi)(\alpha^{n-1}) = 1$

$(\varphi \cup \alpha^{n-1})[M] = 1$

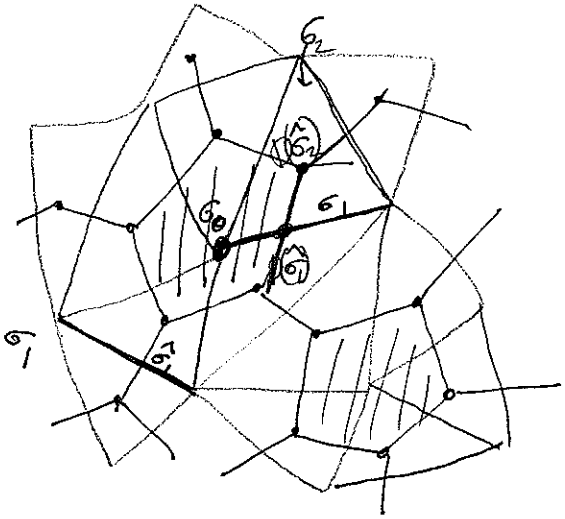
$m\alpha \cup \alpha^{n-1}[M] = 1$

$m\alpha^n = 1 \Rightarrow m = \pm 1$ as desired.

㉔

Poincaré Duality The easy way: [expdiff over Hatcher]

Let M be a closed n-mfld with a triangulation $\mathcal{T}$.



Will construct a dual cell decomp $\mathcal{D}$ where

K simpl in $\mathcal{T}$ $\quad$ n - k cell in $\mathcal{D}$

$$\sigma \longleftrightarrow \hat{\sigma}$$

"reverses inclusion relations"

a sub simplex/cell of

$$\sigma_0 < \sigma_1$$

$$D(\sigma_0) > D(\sigma_1)$$

$n = 2$.

$2 \longleftrightarrow 0$
$1 \longleftrightarrow 1$
$0 \longleftrightarrow 2$

cohomology complex of $\mathcal{T}$

$$C^n \xrightarrow[\text{of chain complexes.}]{\text{isom}} D_n$$

homology complex w.r.t $\mathcal{D}$

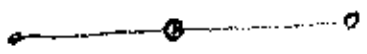
Barycentric subdivision.

$\Delta = [v_0, \ldots, v_n]$ a simplex in $\mathbb{R}^m$

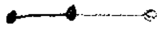
$\delta\sigma_0^*$ has a σ_1^* term $\quad$ $\partial D(\sigma_0)$ has a $\partial D(\sigma_1)$ term

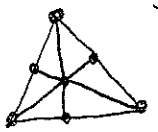
The barycenter of Δ is

$$\hat{\Delta} = \frac{1}{n+1}\sum v_i$$



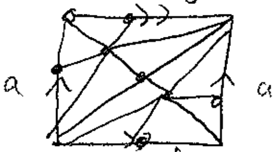
$sd(\cdot) = \cdot$

sd()

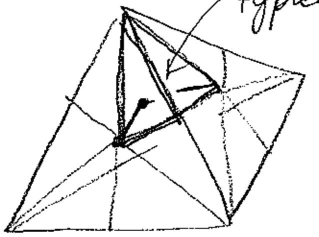
sd()

The barycentric subdivision $sd(\Delta)$ of Δ is

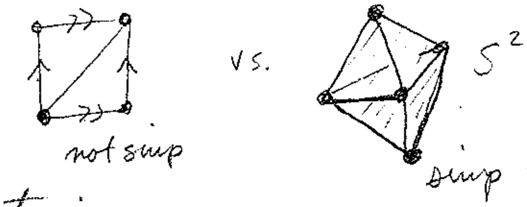
$$\Delta = \bigcup\left\{[w_0, w_1, .., w_{n-1}, \hat{\Delta}] \;\middle|\; [w_0, \ldots, w_{n-1}] \text{ is a simplex in } sd(\partial\Delta)\right\}$$

Can also do to a Δ-complex! 

note how looks like above pict.



Def: A Δ complex is *simplicial* if every set of pts $\{v_0, \dots, v_n\}$ in Δ^0 is the vertices of at *most* one n simplex.

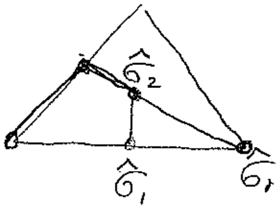


Note any Δ complex can be made simplicial by subdividing *twice*.

Now let J be a triang of M^n. A simplex in sd(J) is of the form $[\hat{\sigma}_1, \hat{\sigma}_2, \dots, \hat{\sigma}_k]$ where

final vertex

init vertex $\hat{\sigma}_1 > \hat{\sigma}_2 > \cdots > \hat{\sigma}_k$



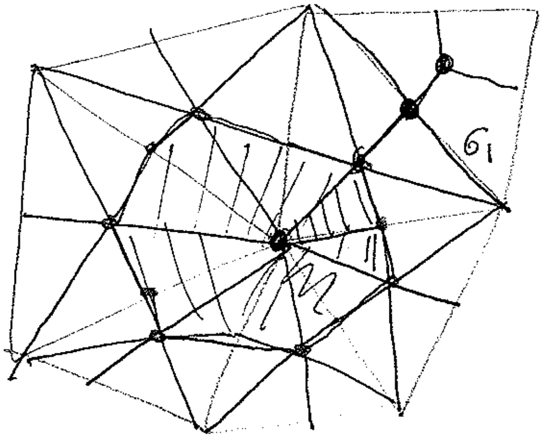
will always order simplices in st(J) this way

Now let σ be a simplex in M

$$D(\sigma) = \text{all open simplices of } sd(J) \text{ w/ } \hat{\sigma} \text{ as the final vertex}$$

$$\overline{D}(\sigma) = \text{closure of } D(\sigma)$$

$$\dot{D}(\sigma) = \overline{D}(\sigma) - D(\sigma)$$

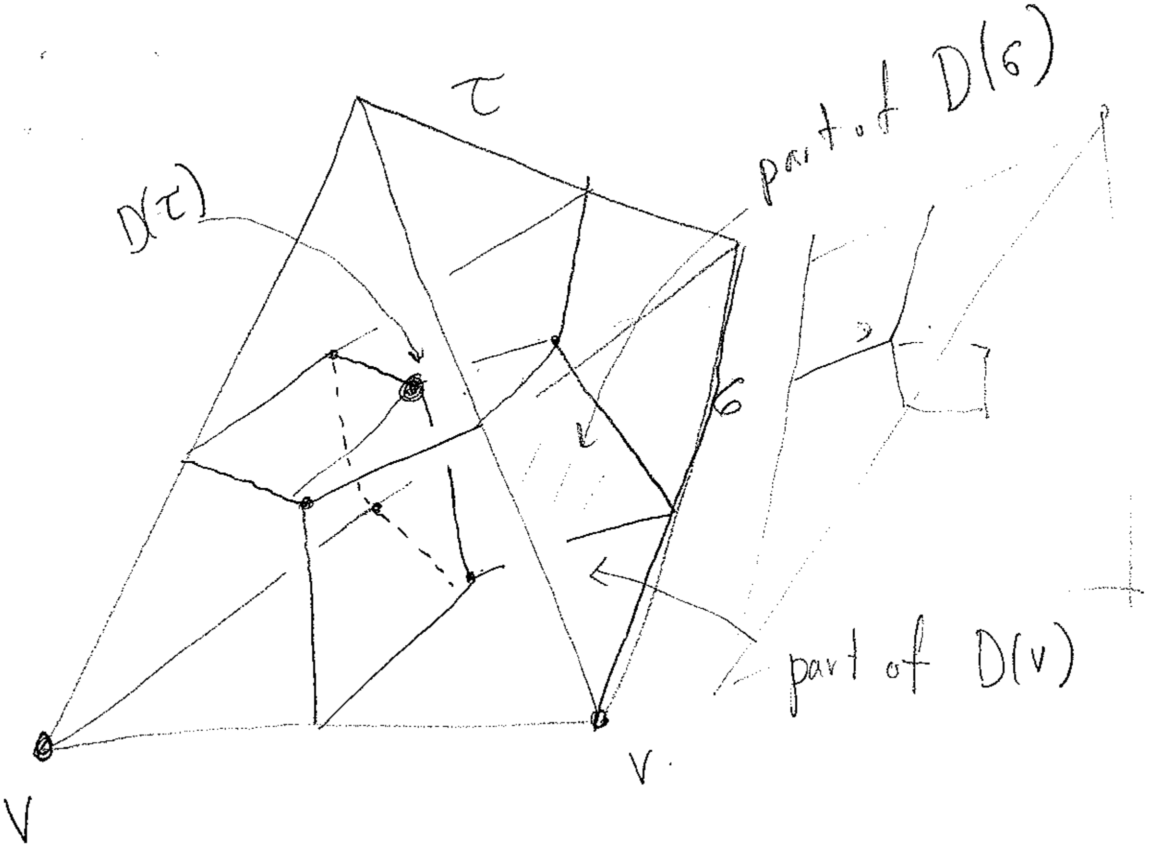


<u>Lemma</u>: (a) The $D(\sigma)$ are disjoint and their union is M.

b) $D(\sigma)$ is a subcmplx of $sd(J)$ of dimension $n-K$ if $|\sigma| = K$.

c) $\dot{D}(\sigma) = \bigcup \{ D(\tau) \mid \tau \gneq \sigma \}$ ← strictly lower dimension

d) $\overline{D}(\sigma) =$ Cone of $\dot{D}(\sigma)$ to $\hat{\sigma}$.

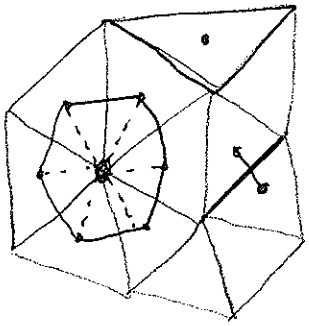


Q: But are the $D(\sigma)$ cells??

Lecture 13: Last time: M an n-manifold w/ triang J.

Constructed the dual complex $\mathcal{D}$

σ a k-simp of J $\longleftrightarrow$ $D(\sigma)$ = all open simp of $sd(\sigma)$ w/ $\hat{\sigma}$ as a final vertex



$\bar{D}(\sigma)$ = union of all such closed cells.

$$\dim(\bar{D}(\sigma)) = n-k$$

Q: Is $(\bar{D}(\sigma), \mathring{D}(\sigma)) \cong (B^{n-k}, \partial B^{n-k})$?

A: Not always! [X Poincaré hom sphere $M = \Sigma^2 X$]

But: $H_*(\bar{D}(\sigma), \mathring{D}(\sigma)) \cong H_*(B^{n-k}, \partial B^{n-k})$ [$\Rightarrow$ using "cellular chains" of $\mathcal{D}$ to compute homology gives right answer.]

Cheat: Will simply assume dual "cells" are really cells. [True in dim 2, 3]

That is we have a "PL" triangulation [which don't always exist.]

Poincaré mod 2: M a cpt conn n-mfld. Then

$$H^k(M;\mathbb{Z}/2) \cong H_{n-k}(M;\mathbb{Z}/2)$$

Pf:

$$\begin{tikzcd}
 & C^{k+1} \arrow[r, "\cong"] & D_{n-k-1} & \\
\text{cochains w.r.t } J & C^k \arrow[u, "\delta"] \arrow[r, "\cong"] \arrow[ur, phantom, "?\circlearrowleft"] & D_{n-k} \arrow[u, "\partial"'] & \text{chains w.r.t. } \mathcal{D} \\
 & \sigma^* \arrow[r, mapsto] & D(\sigma) &
\end{tikzcd}$$

To prove thm, just need to show this commutes.

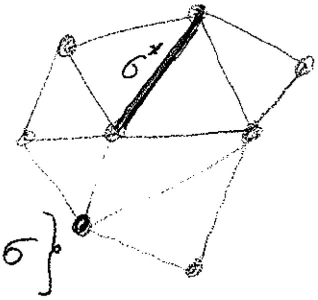
since we're working w/ $\mathbb{Z}/2$ coeffs:

$d \in D_{n-k}$ can be thought of as simply a union of cells

If d is a single cell $D(\sigma)$ $\partial d = \mathring{D}(\sigma) = \bar{D}(\sigma) \setminus D(\sigma)$

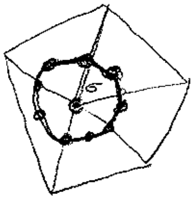
$= \bigcup \{ D(\tau) \mid \tau \gneq \sigma \}$

Similary think of C^* as unions of cells.



$$\delta \sigma^* = \bigcup \{ \tau \mid \tau \geq \sigma \}$$

$\dim \tau = \dim \sigma + 1$



more exactly just the $n-k-1$ cells

$= \bigcup \{ D(\tau) \mid \tau \gneq \sigma \}$

$\dim \tau = \dim \sigma + 1$

Thus the diagram commutes and we're done! ■

What about $\mathbb{Z}$ case? Have to orient things carefully, but same thing works.

orient n simplices of T so that they form a cycle,

$\left[\sum \sigma_i \right] = [M] \in H_n(M;\mathbb{Z}) = \mathbb{Z}$.

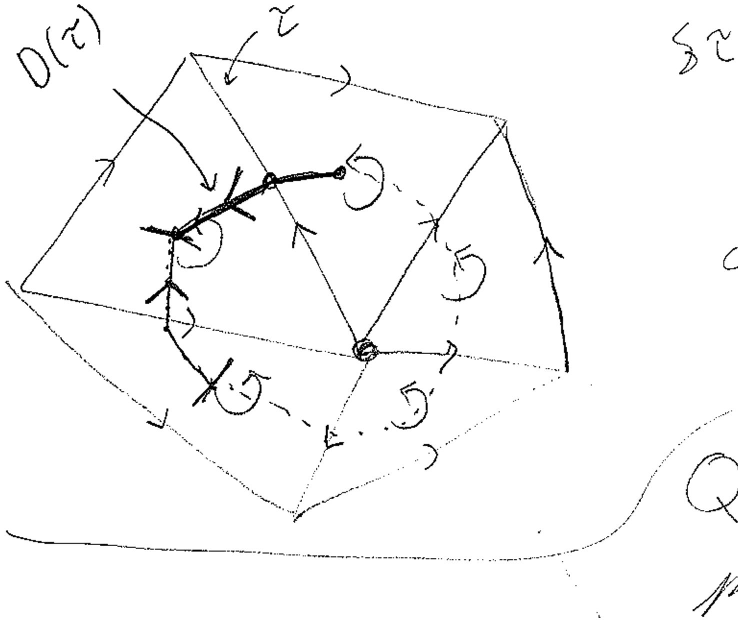
σ_i an n-simplex

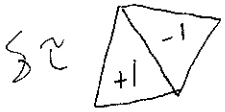
$$\begin{tikzcd}
\sigma_i^* \arrow[r, mapsto] & D(\sigma_i) \\
C^n \arrow[r] & D_0 \\
C^{n-1} \arrow[r] \arrow[u, "\delta"] & D_1 \arrow[u, "\partial"'] \\
\tau^* \arrow[r, mapsto] & D(\tau)
\end{tikzcd}$$

$$\delta \tau^* = \partial D(\tau) = \pm D(\sigma_i) \pm D(\sigma_j)$$

$$\| $$

$$\pm \sigma_i^* \pm \sigma_j^*$$





Inductively push down to finish job.

Q: Where does the cap product come in?

See ex 3.31 in text, or Munkres "Elements of diff top" §67.

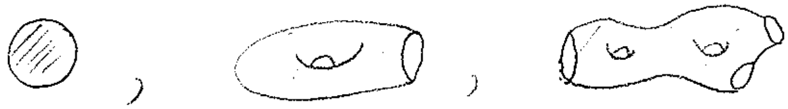
Other forms of duality:

Lecture 14: Today: 1) Other forms of Poincaré Duality
2) Intro to higher homotopy groups.

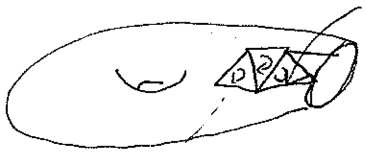
Other forms of duality:

M a mfld w/ boundary, i.e. every $x \in M$ has an open nbhd homeo to $\mathbb{R}^n$ or $\{x \in \mathbb{R}^n \mid x_n \geq 0\}$.

$n=2$



Note: $H_n(M;\mathbb{Z}) = 0$ when ∂M is non-empty



M orientable $\Longleftrightarrow$ $H_n(M,\partial M;\mathbb{Z}) \cong \mathbb{Z}$.

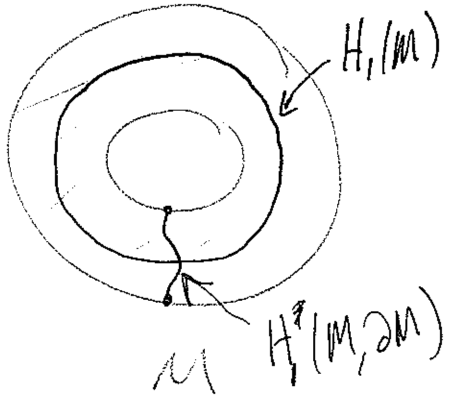
Lefschetz duality: Suppose M is a cpt con n-mfld, poss w/ ∂. Then if M is orientable, then

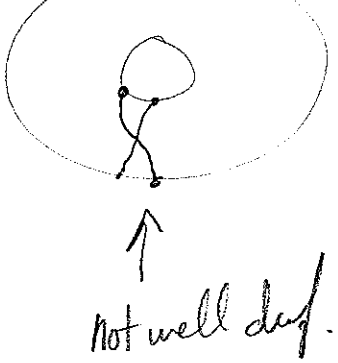
$$H^k(M;\mathbb{Z}) \cong H_{n-k}(M,\partial M;\mathbb{Z})$$
$$H^k(M,\partial M;\mathbb{Z}) \cong H_{n-k}(M;\mathbb{Z})$$

[again mediated by cap prod w/ the fund class]

Geometric picture:

$$\mathbb{Z} = H^1(M) = \text{Hom}(H_1(M)) \cong H_1(M,\partial M)$$





Alexander Duality: K a cpt, locally contractible nonempty proper subset of S^n then

$$\tilde{H}_i(S^n \setminus K) \cong \tilde{H}^{n-i-1}(K;\mathbb{Z}) \quad \text{for all } i.$$

[Point: doesn't matter how K is embedded!]

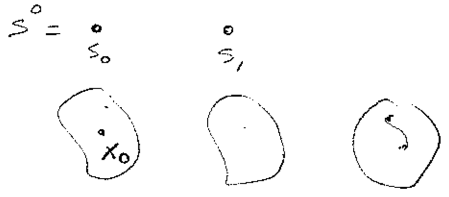
* * *

Higher homotopy groups: Fix a basept s_0 in S^n.

$\pi_n(X, x_0) =$ Homotopy classes of maps

$$(S^n, s_0) \longrightarrow (X, x_0)$$

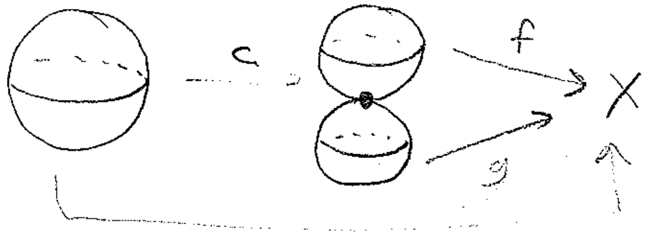
[where all maps in the homotopy are of this form.]

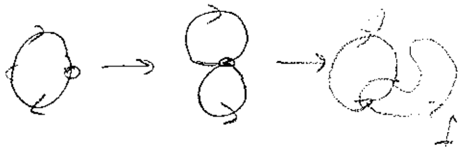


$\pi_0 =$ set of path comp of X

$\pi_1 =$ fund gp

π_n, $n \geq 2$ higher hom. gps.



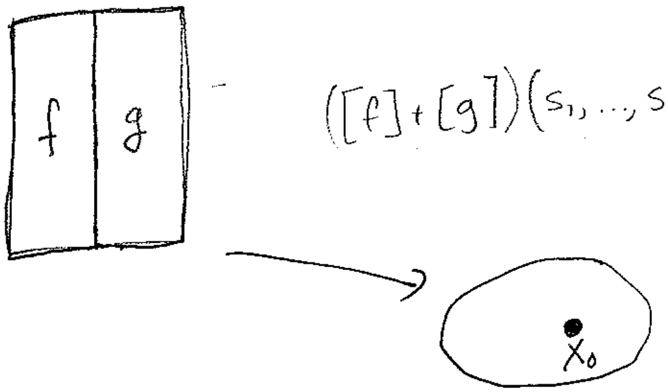


$[f] + [g]$

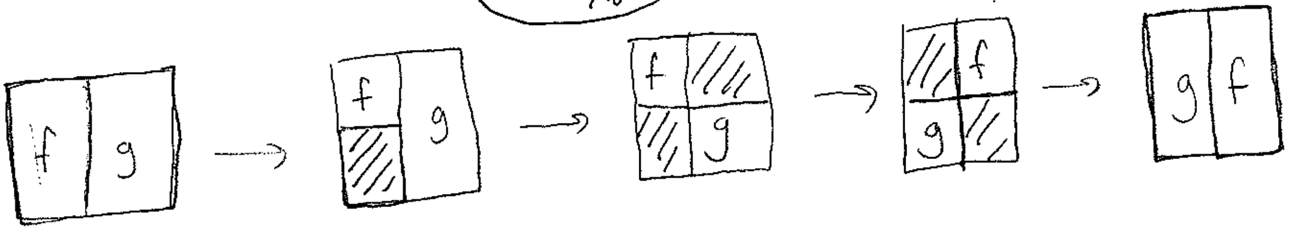
Blather about how hard to compute, etc.

Prop: π_n is abelian.

Pf: Think of $\pi_n = \begin{smallmatrix}\text{hom}\\ \text{classes}\\ \text{of}\end{smallmatrix} \; (I^n, \partial I^n) \longrightarrow (X, x_0)$



$$([f]+[g])(s_1,\dots,s_n) = \begin{cases} f(2s_1, s_2,\dots,s_n) & s_1 \le 1/2 \\ g(2s_1-1, s_2,\dots,s_n) \end{cases}$$



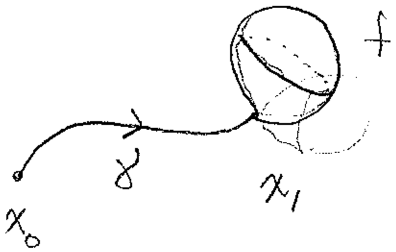
■

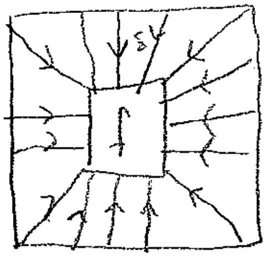
Prop: If X is path connected then $\pi_n(X, x_0)$ doesn't depend on x_0.

Pf: Let γ be a path in X joining x_0 to x_1

$f : (I, \partial I) \longrightarrow (X, x_1)$

$\gamma f : (I, \partial I) \longrightarrow (X, x_0)$



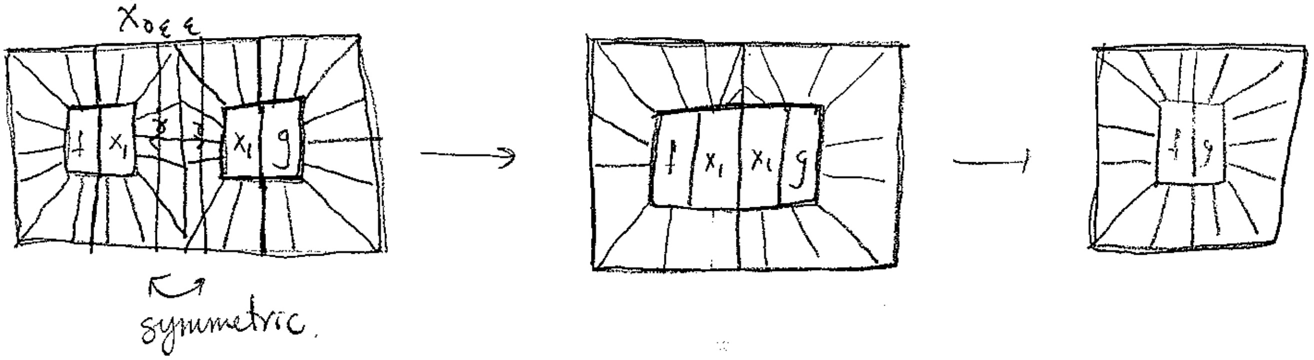


induces a map

$$\pi_1(X, x_1) \longrightarrow \pi_1(X, x_0)$$

which is a homomorphism

$\gamma f + \gamma g$



also have $\pi_n(X, x_0) \longrightarrow \pi_n(X, x_1)$ given by γ^{-1}

These are inverse isomorphisms ▨

[Sometimes drop the base pt.]

π_n is a functor:

$\varphi : (X, x_0) \longrightarrow (Y, y_0)$ induces $\pi_1(X, x_0) \xrightarrow[\varphi_*]{} \pi_1(Y, y_0)$
given by
$f : (S^n, s_0) \to (X, x_0) \longmapsto f \circ \varphi.$

Lemma:

Lecture 15: Last time: $\pi_n(X, x_0) =$ hom classes of maps $(S^n, s_0) \to (X, x_0)$.

Today: Some basic properties.

$$\begin{tikzcd} (X, x_0) \arrow[r, "\varphi"] & (Y, y_0) & \text{get } \varphi_*: \pi_n(X, x_0) \arrow[r] & \pi_n(Y, y_0) \\ & & f \arrow[u, phantom, "\in" rotate=90] \arrow[r, mapsto] & f \circ \varphi \end{tikzcd}$$

Note deps only on the base pt preserving homotopy class of φ.

Prop: $\varphi : (X, x_0) \to (Y, y_0)$ is a homotopy equiv. Then φ_* is an isom on π_n for all n.

Cor: If X is contractible then $\pi_n X = 0$ for all $n \geq 1$

[as $\pi_n(\{x_0\}, x_0) = 0$]

Prop: A covering map $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ induces $\cong$ on π_n for $n \geq 2$.

Pf: Consider $\pi_n(\tilde{X}, \tilde{x}_0) \xrightarrow{p_*} \pi_n(X, x_0)$

onto: As $\pi_1 S^n = 1$, we always have a lift.

$$\begin{tikzcd} & (\tilde{X}, \tilde{x}_0) \arrow[d, "p"] \\ (S^n, s_0) \arrow[ur, dashed] \arrow[r, "f"'] & (X, x_0) \end{tikzcd}$$

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ S^n \arrow[ur, "g"] \arrow[r, "g \circ p"] & X \end{tikzcd}$$

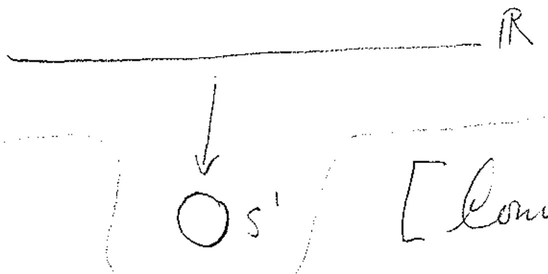
1-1: Suppose $g \in \pi_n(X, x_0)$.

If $g \circ p = 0$, then $g \circ p \simeq$ const map $\Longrightarrow$ by covering homotopy prop. $g = 0$ in $\pi_n(X, x_0)$ ▨

[Aside: $f = 0$ in π_n iff it extends over B^n.]

Ex: $\pi_n(S^1) = 0$ for all $n \geq 2$. $\pi_n(T^k) = 0$ for all $n \geq 2$.

[Why.]



[Compare: Kunneth formula!]

Thm: X_α are path conn. Then $\pi_n(\prod_\alpha X_\alpha) = \prod_\alpha \pi_n(X_\alpha)$

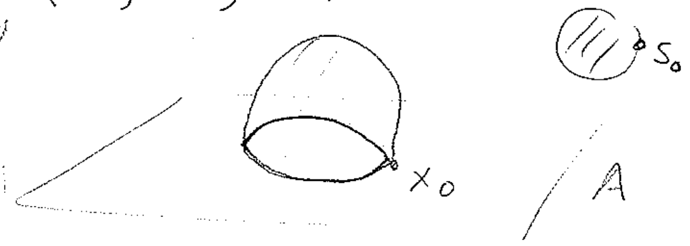
Pf: $f : S^n \longrightarrow \prod_\alpha X_\alpha$ is the same as $\{f_\alpha : S^n \to X_\alpha\}$

a homotopy between such is the same as $\{F_\alpha : S^n \times I \to X_\alpha\}$ ▨

Note can manipulate coor independently.

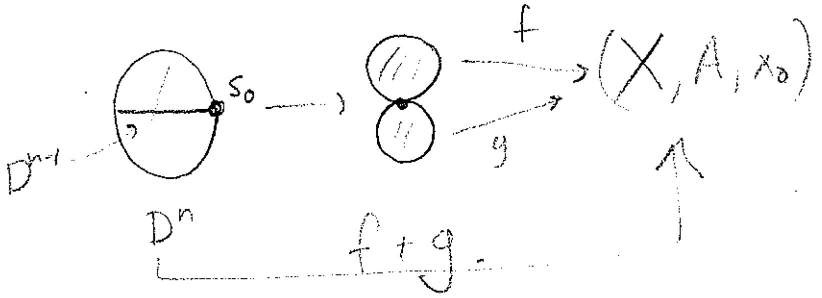
Relative homotopy groups: $X \supseteq A \ni x_0$

$\pi_n(X, A, x_0) =$ homotopy classes of maps $(D^n, S^{n-1}, s_0) \longrightarrow (X, A, x_0)$



Have group opp

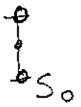
$$D^n \xrightarrow[D^{n-1}]{\text{crush}} D^n \vee D^n \longrightarrow X$$

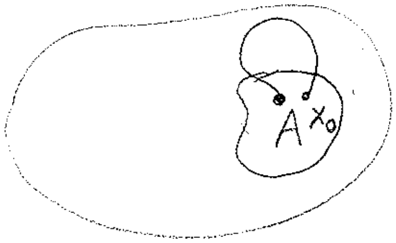


Taking $A = \{x_0\}$ gives usual $\pi_n(X, x_0)$

Special cases: $n = 0$ doesn't really make sense

$n = 1$:





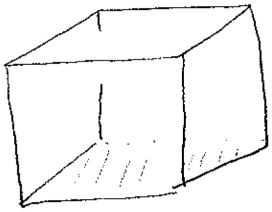
× $\pi_1(X, A, x_0)$ is hom classes of paths w/ one end at x_0 and other in A.

No group law

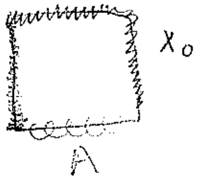
$n = 2$: group str may not be abelian.

Prop: $n \geq 3$ $\pi_n(X, A, x_0)$ is abelian

Alt description:

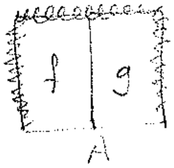


$$(I^n, \partial I^n, J) \longrightarrow (X, A, x_0)$$

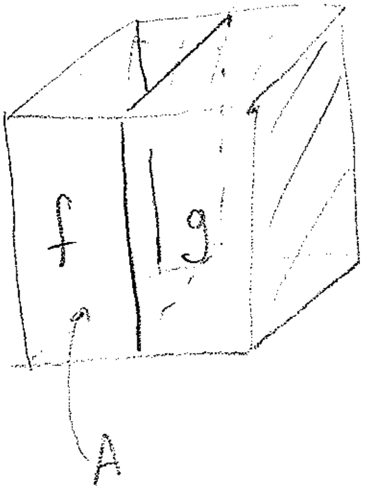


$I^n \supseteq I^{n-1} = \{s_n = 0\}$ $J = \partial I^n \setminus \mathrm{int}(I^{n-1})$

$f + g =$



I^{n-1} is special, as doesn't go to base pt.

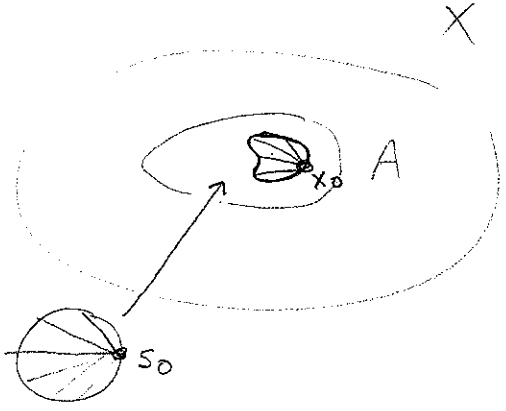


However comm trick still works w/ $n = 3$.

Note: [Compression criterion]

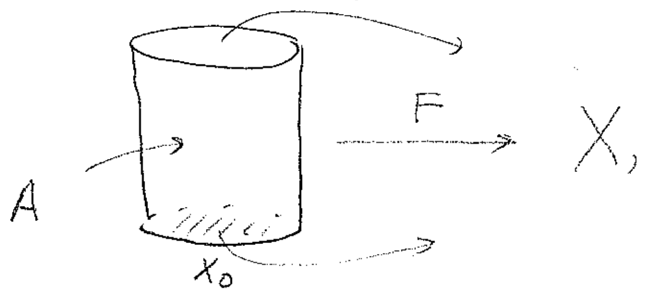
$f:(D^n,S^{n-1};s_0)\to(X,A,x_0)$ is 0 in π_n

iff it is homotopic, rel S^{n-1}, to a map with image in A.

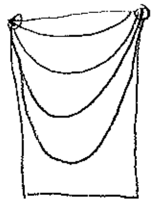


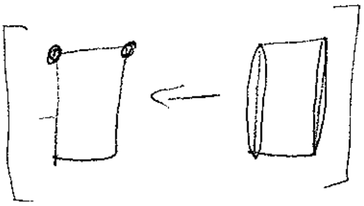
Pf: (⇐) If $f(D^n)\supseteq A$, then can homotope to const map.

(⇒) suppose $f=0$ in π_n. Thus



Now consider this fam of discs





Long exact sequence:

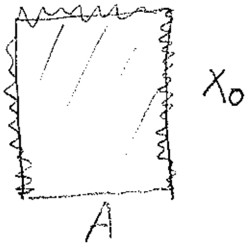
$$\to \pi_n(A,x_0)\xrightarrow{i_*}\pi_n(X,x_0)\xrightarrow{j_*}\pi_n(X,A,x_0)\xrightarrow{\partial}\pi_{n-1}(A,x_0)\to$$

$$\cdots\to\underbrace{\pi_1(X,A,x_0)\to\pi_0(A,x_0)\to\pi_0(X,x_0)}$$

not groups, but exactness still makes since, as these have a dist. elt namely the const map.

$$\partial\big(f:(I^n,\partial I^n,J)\to(X,A,x_0)\big)=f|_{I^{n-1}}$$

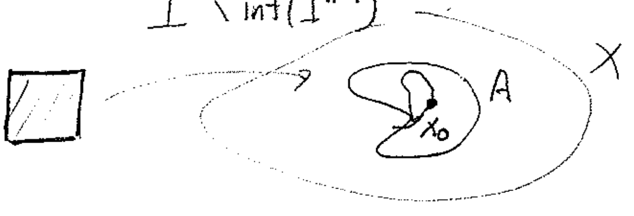
$$(D^n,S^{n-1};s_0)\to(X,A,x_0)\big)\qquad f|_{S^{n-1}}$$



Lecture 16: Last time: Relative homotopy groups. MIDTERM

$$\pi_n(X,A,x_0) = \text{hom classes of maps } (I^n, \partial I^n, J^{n-1}) \xrightarrow{f} (X,A,x_0)$$

$$J^{n-1} = I^n \setminus \mathrm{int}(I^{n-1})$$



Long exact sequence:

[induced by inclusion]

$$\to \pi_n(A,x_0) \xrightarrow{i_*} \pi_n(X,x_0) \xrightarrow{j_*} \pi_n(X,A,x_0) \xrightarrow{\partial} \pi_{n-1}(A,x_0) \to \cdots$$

$$\to \underbrace{\pi_1(X,A,x_0) \to \pi_0(A,x_0) \to \pi_0(X,x_0)}$$

[not gps, but have disting. elts (the const map to x_0) so exactness still makes sense]

Where: $\partial f = f|_{I^{n-1}}$

Pf: [Special cases are an exercise.]

Exactness at $\pi_n(X,x_0)$: Note $j_* \circ i_* = 0$ by comp. criterion.

$\Rightarrow \mathrm{Im}\, i_* \subseteq \mathrm{Ker}\, j_*$. Suppose $j_*(f) = 0 \Rightarrow f$ can

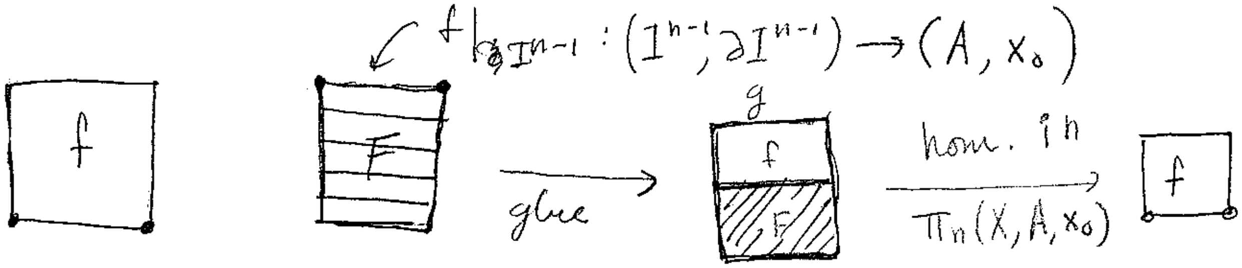
be hom into A keeping $f|_{\partial I^n}$ fixed. Thus $f \in \mathrm{Im}(i_*)$

Exactness at $\pi_n(X,A,x_0)$: $\partial \circ j_* = 0$ as $\partial(\square) =$

f rest to ↑

For $\mathrm{Im}\, j_* \supseteq \mathrm{Ker}\, \partial$, suppose $f : (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$

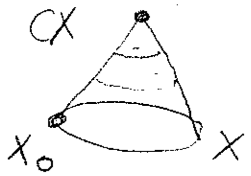
has $f|_{I^{n-1}} = 0$ in $\pi_n(A, x_0)$. Let F be the homotopy from $f|_{I^{n-1}}$ to const.



Thus $j_*([g]) = [f]$ as desired. $\overset{\uparrow}{\pi_n(X, x_0)}$ $\boxed{\text{Remaining case similar, ex}}$ ▨

<u>Ex</u>: $CX = X \times [0,1] \big/$ collapse $X \times \{0\}$ to a pt

$\begin{array}{l} CX \\ \cup\!\mid \\ X \text{ as } X \times \{1\} \end{array}$



$$\begin{array}{ccccccc} \pi_n(CX) & \to & \pi_n(CX, X) & \xrightarrow{\cong} & \pi_{n-1}(X) & \to & \pi_{n-1}(CX) \\ 0 & & & & & & 0 \end{array}$$

[provided $n-1 > 0$]

In particular, can make $\pi_2(CX, X)$ any group by choosing $\pi_1(X)$ appropriately.

Thm: $f: X \to Y$ a map of connected CW complexes. If f_* is an isom on all π_n then f is a homotopy equiv.

<u>Compression Lemma</u>: (X, A) a CW pair, (Y, B) any pair w/ $B \neq \emptyset$. For each n such that $X \setminus A$ has a cell of dim n, assume that $\pi_n(Y, B, y_0) = 0$ for <u>all</u> $y_0 \in B$. Then every $f: (X, A) \to (Y, B)$ is homotopic rel A to a map $X \to B$.

Note: when $n=0$, the cond should be interpreted as saying (Y,B) is 0-connected, i.e. path connected.

Pf: Assume by induction that $f(X^{k-1}) \subseteq B$

$$\Phi : D^k \to X \text{ the char map of a cell } e^k \text{ of } X - A$$

Then

$$f\circ\Phi : (D^k, \partial D^k) \to (Y,B)$$

Is homotopic, rel ∂D^k into B

Do for all k cells in $X \setminus A$ at once ~~#~~,
~~const com~~ gives a hom of $f|X^k \cup A$ ~~$\to B$~~ to a map into B.

By homotopy ext prop this extends to a ~~map of~~ homtopy to a map of all of X unchanged on ~~B~~ X^k.

In general preform homotopy on time $[1-2^{-k}, 1-2^{-k+1)}]$.

Lecture 17 | Last time:

Compression Lemma: (X,A) a CW pair, (Y,B) any pair w/ $B \neq \emptyset$. For each n where there are n-cells in $X \setminus A$, assume that $\pi_n(Y, B, y_0) = 0$ for all $y_0 \in B$. Then every map $f: (X,A) \longrightarrow (Y,B)$ is hom, rel A, to a map $X \to B$.

Whitehead's Thm: $f: X \longrightarrow Y$ a map between connected CW complexes. If f_* induces $\cong$ on all π_n, then f is a homotopy equiv. If f is the inclusion of a subcomplex, then X is a deformation retract of Y.

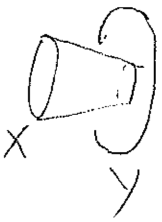
Cor: If Y is a CW complex w/ $\pi_n(Y) = 0$ for all n, then X is contractible.

Pf: Take $X = \{pt\}$ in Y.

Pf of Thm: First, consider the case when X is a subcomplex. From the long exact seq, we get

$$\begin{tikzcd}
\pi_n(X) \arrow[r, "\cong"] & \pi_n(Y) \arrow[r] \arrow[r, bend left, dashed, "0"] & \underbrace{\pi_n(Y,X)}_{0} \arrow[r] \arrow[r, bend left, dashed, "0"] & \pi_{n-1}(X) \arrow[r, "\cong"] & \pi_{n-1}(Y)
\end{tikzcd}$$

So the compression lemma applied to $(Y,X) \xrightarrow{id} (Y,X)$ gives the needed def. retraction.

Recall: $M_f =$  $(X\times[0,1] \sqcup Y)\big/_{(x,1)\sim f(x)}$

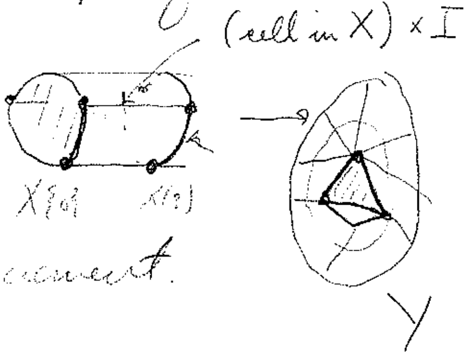
Mapping cylinder.

Note: M_f def retracts to Y; $$\begin{tikzcd} X \arrow[r, hook] \arrow[rr, bend left, "f"] & M_f \arrow[r, "\text{ret}", "\simeq_{h.e.}"'] & Y \end{tikzcd}$$

Claim: M_f def ret to X.

If f is cellular, i.e. $f(X^k) \subseteq Y^k$ for all k, then

M_f is a CW complex



so we can apply the easier argument.

In general we can use the following to reduce to this case. ▨

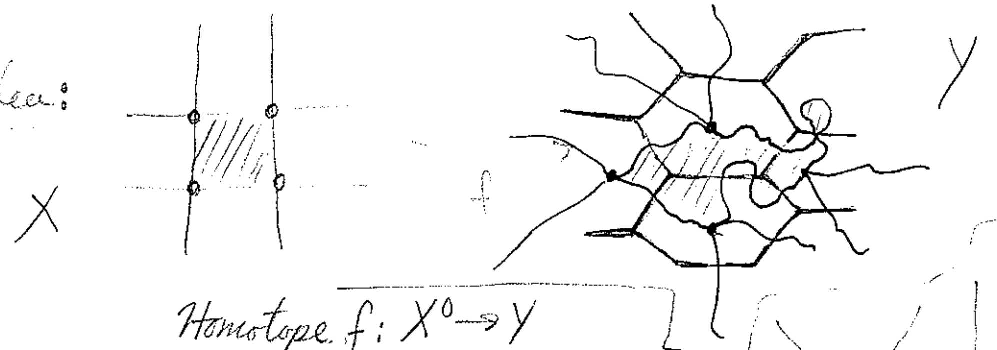
Thm: Every map $f: X\to Y$ of CW complexes. Then f is homotopic to a cellular map. If f is already cellular on a subcomplex A, then the hom is stationary on A.

Cor: $\pi_n(S^k) = 0$ for $n < k$.

Pf: Consider usual cell decomp of S^n, S^k w/ one 0-cell and only one other cell. Then any map $S^n \longrightarrow S^k$ w/ $n<k$ is homotopic to the const map.

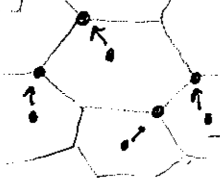
[Query: How did you show $\pi_1 S^2 = 1$?]

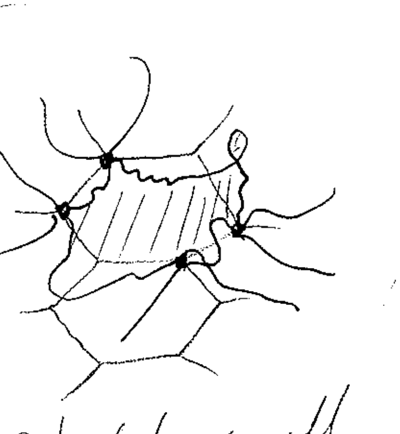
Idea:



Homotope $f: X^0 \to Y$

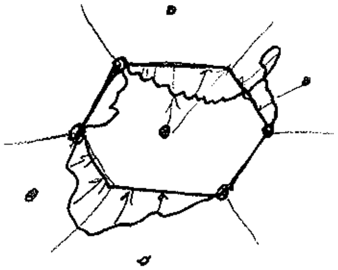
Proceed inductively: so that $f(X^0) \subseteq Y^0$ is easy. Extend to all of X by Prop 0.16.





extended over all of X.

1) Homotope $f: X^1 \to Y$, rel X^0, so that $f(X^1) \subseteq Y^1$ by picking p_i in each cell of $Y \setminus Y^1$ not in image and pushing outward.



Again extend to all of X

n) Repeat for each n in the same manner.

[Query: What is missing here.]

Lemma: $Z = (W$ w/ a k cell $e^k)$, $f: I^n \to Z$ w/ $n < k$. Then f is homotopic, rel $f^{-1}(W)$, to a map g where $g(I^n)$ is a proper subset of $\mathrm{int}(e^k)$.

Lecture 18: Today: Actual computations!

Def X is n-connected if $\pi_i(X, x_0) = 0$ for all $i \leq n$

[0-conn = path conn, 1-conn = simply conn; equiv to every map $S^i \to X$ is hom to a const map for $i \leq n$]

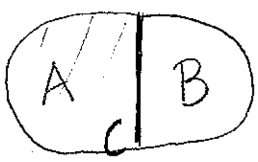
Def: (X, A) is n-connected if $\pi_i(X, A, x_0) = 0$ for all $x_0 \in A$ and $0 < i < n$ and each path comp of X contains a pt of A. [$\pi_i = 0 \iff$ every map $(D^i, \partial D^i) \to (X, A)$ is hom, rel ∂D^i into A]

[Comment on central nature of excision $H_n(X,A) \cong \tilde{H}_n(X/A)$]

Thm: Let X be a CW complex decomposed as the union of two subcomplexes A and B w/ $C = A \cap B \neq \emptyset$.

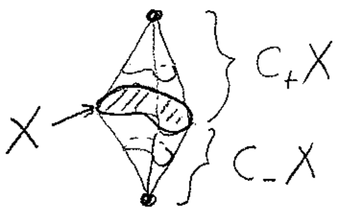
If (A, C) is m-connected, (B, C) n-connected, then

$\pi_i(A, C) \xrightarrow{i_*} \pi_i(X, B)$ is an isomorphism for $i < n+m$
a surjection for $i = n+m$.

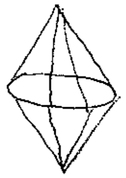


[Compare w/ homology statement]

$$SX = X \times [-1, 1] \Big/ \begin{matrix} X \times \{1\} \\ X \times \{-1\} \end{matrix}$$



$SS^n = S^{n+1}$



S functor $f : X \to Y$

$Sf : SX \to SY$

$f \times (\text{id on } [-1,1])$

$$\pi_n(X, x_0) \xrightarrow{S} \pi_{n+1}(SX, x_0 \times 0)$$

Freudenthal suspension thm: Suppose X is $n-1$ connected

Then $\pi_i X \xrightarrow{S} \pi_{i+1} SX$ is an isom for $i < 2n-1$
and a sur for $i = 2n-1$.

Cor: $\pi_n S^n \cong \mathbb{Z}$ and is gen by the identity map.
The homotopy class of $f: S^n \to S^n$ is determined by its degree.

Pf of Cor: Consider the suspension induced maps

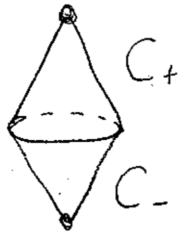
via F.S.T as S^n is $(n-1)$ connected.

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "\cong"] & \pi_1 S^1 \arrow[r, "\boxed{\text{onto}}"] \arrow[dr, "\text{degree onto}"'] \arrow[r, phantom, shift right=4ex, "\circlearrowright"] & \pi_2 S^2 \arrow[r, "\cong"] \arrow[d, "\text{degree onto}"] & \pi_3 S^3 \arrow[r, "\cong"] & \pi_4 S^4 \arrow[r] & \cdots \\
\text{gen by id} & & \mathbb{Z} & & &
\end{tikzcd}$$

$\Longrightarrow$ first map is injective.
degree determines homotopy type.

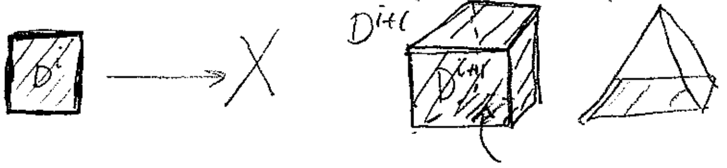
$f: S^2 \to S^2 \qquad H_2(S^2) \circlearrowleft f_* \qquad f_*(c) = \deg(f)\, c$. ▨

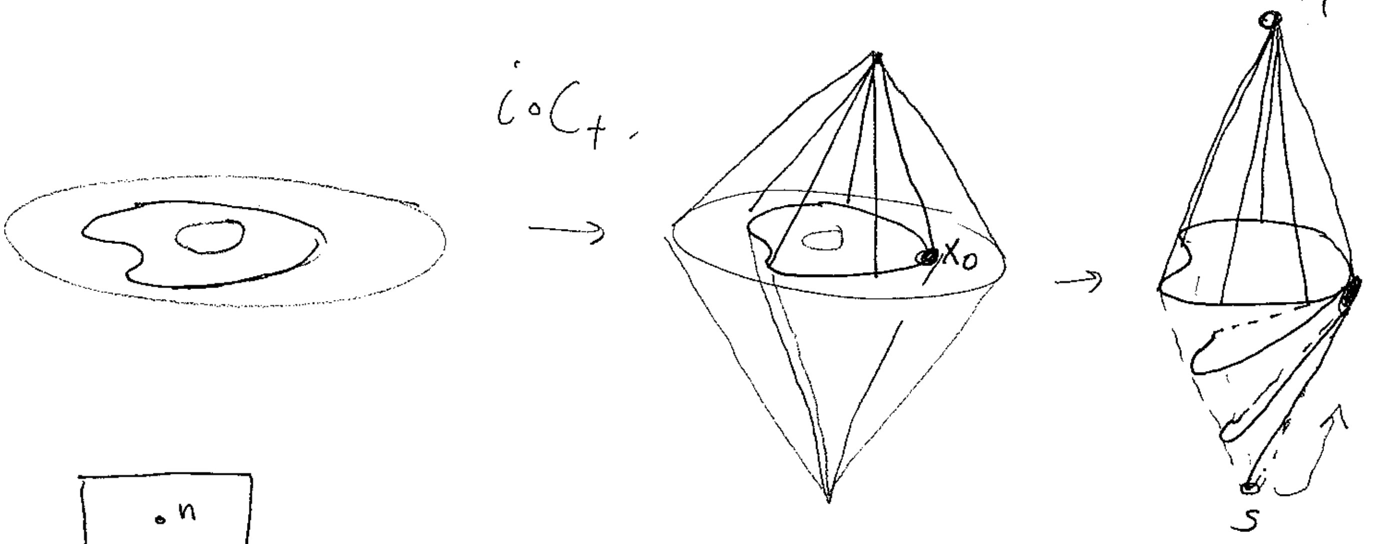
Pf of F.S.T from excision: First observe $SX \cong$

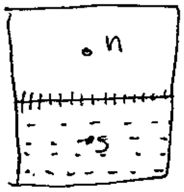


$$\begin{tikzcd}
\pi_i(X) \arrow[r, bend left, "C_+"] \arrow[rrr, bend right, "S_*"'] & \pi_{i+1}(C_+X, X) \arrow[l, "\partial"', "\cong"] \arrow[r, "i_*"] & \pi_{i+1}(SX, C_-X) & \pi_{i+1}(SX) \arrow[l, "\cong"]
\end{tikzcd}$$

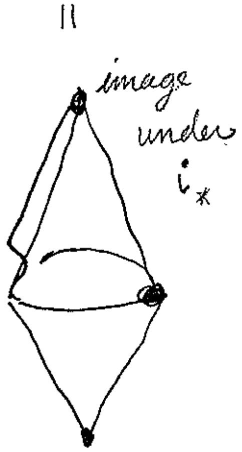
$S^i \to X \longmapsto \underset{\underset{D^{i+1}}{\cong}}{CS^i} \to C_+X$







$$\pi_{i+1}(C_+X, X) \to \pi_{i+1}(SX, C_-X)$$



X $(n-1)$ connected $\Rightarrow$ (C_+X, X) n connected

$X = X \cap C_-X$ $n-1$ connecte

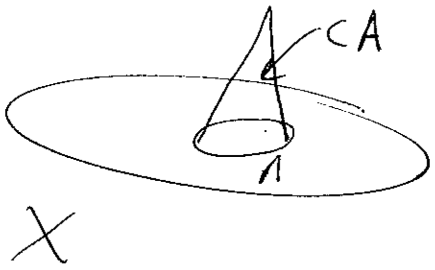
$\Rightarrow$ map is an isom for $i+1 < 2n$
onto for $i+1 = 2n$

▨

$\pi_n(\vee_\alpha S^n_\alpha)$ is free allelian on obvious generators.

Prop: (X,A) a r-connected pair, A s-connected

then $\pi_i(X,A) \to \pi_i(X/A)$ is an isom $i \le r+s$
and a surjection $i = r+s+1$



$X \cup CA/$~~CA~~ $\simeq$ h.e. to $X \cup CA/CA$

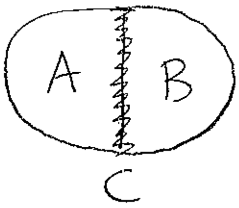
$\simeq X/A$.

$$(X,A) \longrightarrow (X \cup CA, CA)$$

Lecture 19: Write up at beginning

Thm: X a CW complex which is the union of subcomp. A, B where $C = A \cap B \neq \phi$. If (A,C) is n-conn, (B,C) m-conn Then $\pi_i(A,C) \longrightarrow \pi_i(X,B)$ is isom for $i < n+m$

a surjection $i = n+m$

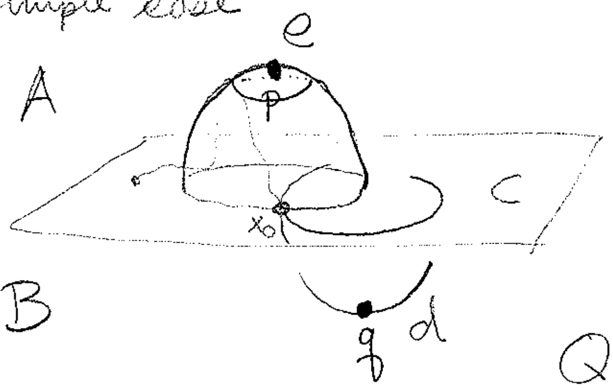


"Excising B\C"

Pf: [Will do some special cases]

Note if $A \setminus C$ has no n cells, then (A,C) is n-connected.

Simple case



$A = C \cup e^2$ $B = C \cup d^2$

everything is 1-connected

(X,B) is also 1-connected

Q: Why is $\pi_2(A,C) \to \pi_2(X,B)$ onto?

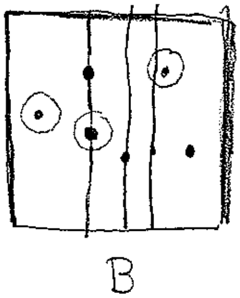
Consider: f : 

$\longrightarrow X$. Consider $p \in e$, $q \in d$

homotope f to be "nice" i.e. $f^{-1}(p), f^{-1}(q)$ = finite number of points.

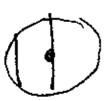
map locally a "linear" homeo near these points.

Consider



Claim: Can choose p so that no point of p, q lie on same vertical line

Reason: Image of lines has too low a dimension



So now we have

choose $\varphi : I \to I$ so all of $f^{-1}(q)$ lie below the graph, $f^{-1}(p)$ lie above it, $\varphi = 0$ on ∂I

Consider

$$\begin{tikzcd}
\pi_2(A,C) \arrow[r] \arrow[d, "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & \pi_2(X,B) \arrow[d, "\cong"] & \text{by homotopy equiv} \arrow[l] \\
\pi_2(X \setminus q, X \setminus \{p,q\}) \arrow[r] & \pi_2(X, X \setminus p) &
\end{tikzcd}$$

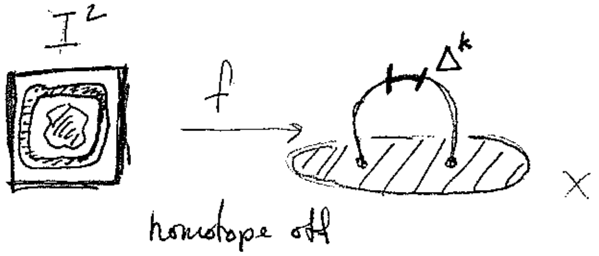
f

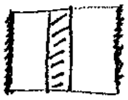
In $\pi_2(X, X\setminus p)$, f is homotopic to

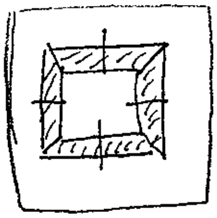
which is in $\pi_2(X\setminus q, X\setminus\{p,q\})$. Thus the horizontal maps are surjective, as desired.

Now suppose e has dim $n+1$ and d has dim $m+1$

<u>Lemma 4.10</u>: Let $f : I^n \to Z = W \cup e^k$. Then f is homotopic, rel $f^{-1}(W)$ to a map f_1 s.t. $\exists$ a simplex $\Delta^k \subseteq e^k$ where $f^{-1}(\Delta^k)$ is a union of finitely many convex polyhedra on each of which f_1 is the restriction of a linear surjection $\mathbb{R}^n \to \mathbb{R}^k$







Some examples

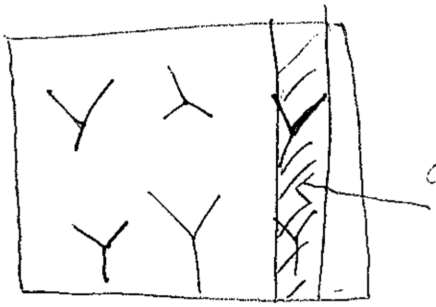
← a more confusing one.

Do this for. out map f w.r.t simplices $\Delta^{n+1}_e \subseteq e$

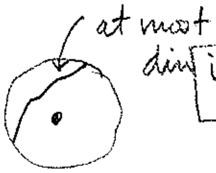
$\Delta^{m+1}_d \subseteq d$

now $f^{-1}(p)$ is finite union of poly of dim $i-n-1$

$f^{-1}(q)$ is finite . . . dim $i-m-1$

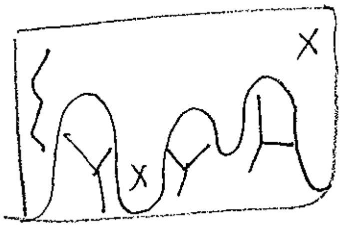


Claim: Can choose p so that no pt of p, q are on same vertical line.



As long as $\boxed{i-m < n+1}$ we can perturb p so that the claim holds

So



as before. Hence crucial cond is $i \leq n+m$.

1-1 is similar but w/ homotopys so you lose one dimension..

In general case, first consider when $A = C \cup$ many $n+1$ cells. Then $B = C \cup$ many $m+1$ cells. When A has cells of dim $> n+1$, use induction on skeleta.

Cor Prop 4.13: X is homotopy equiv to X' (fixing C) s.t. $A = C \cup$ cells of dim $> n$
$B = C \cup$ cells of dim $> m$.

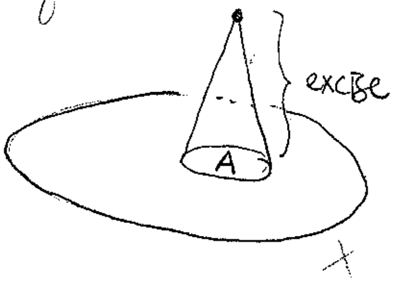
Lecture 20: Last time: Pf of excision ③⑨

Today: Eilenberg-MacLane Spaces $\boxed{H_n(X,A) \cong \tilde{H}_n(X/A)}$

Prop: Suppose (X,A) is a r-connected CW pair, A is s-connected, then $\pi_i(X,A) \to \pi_i(X/A)$ induced by the quotient map $X \to X/A$ is an isomorphism for $i \leq r+s$ and a surjection for $i = r+s+1$

[Basic idea turn maps into spaces.]

Pf: $X \cup CA \simeq_{h.e.} X/A$ (Prop 0.17).



$$\begin{tikzcd}
\pi_i(X,A) \arrow[r] & \pi_i(X \cup CA, CA) \arrow[r, "\cong"] & \pi_i(X \cup CA/CA) \\
& \pi_i(X \cup CA) \arrow[u, "\cong"] \arrow[ur, "\cong"] &
\end{tikzcd}$$

$\cong \pi_i(X/A)$

long exact

as homotopy equivalence. ▨

excision noting (CA,A) is $s+1$ connected.

Eilenberg-MacLane Spaces: If $\pi_n(X) = G$ and all other $\pi_i(X) = 0$ call X a $K(G,n)$.

Ex: S^1 is a $K(\mathbb{Z},1)$, $T^n = \mathbb{R}^n/\mathbb{Z}^n$ is a $K(\mathbb{Z}^n,1)$

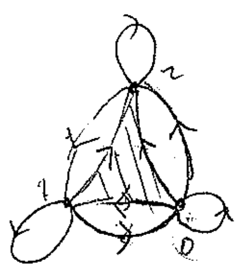
In general, a CW complex X is $K(G,1)$ w/ $G = \pi_1 X$ iff its universal cover is contractible. E.g. X = closed orient surface of genus >0.

Ex: $\mathbb{R}P^\infty$ is a $K(\mathbb{Z}/2,1)$ [Query: What is universal cover? S^∞]

Ex: $\mathbb{C}P^\infty$ is a $K(\mathbb{Z},2)$ [it turns out. Thought it is clear that $\pi_2 \mathbb{C}P^\infty = \mathbb{Z}$]

Thm: For any group G, there is a CW complex which is a $K(G,1)$. If G is abelian, then for all n, there is a $K(G,n)$. These spaces are unique up to homotopy equivalence.

$$\begin{tikzcd} \tilde{X} \arrow[d] & \leftarrow \text{contractible, free } G \text{ action} \\ X & \end{tikzcd}$$



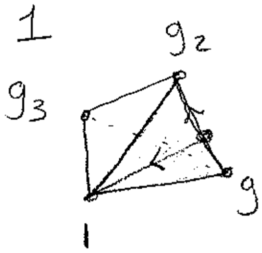
$G = \mathbb{Z}/3$

Pf of existence: Case $n=1$:

EG = Δ-complex consisting of an n-simplex for every ordered $n+1$ tuple $(g_0, g_1, \ldots, g_n)$ [Glued by obvious res. relation.]

[vertices $\leftrightarrow G$

Contractible: Given $x \in \Delta_{(g_0,\ldots,g_n)}$ homotope to 1 via the straight line in $\Delta_{(1,g_0,\ldots,g_n)}$



[Doesn't matter which simplex we regard x as lying in.]

G acts freely on EG via $\Delta_{(g_0,g_1,\ldots,g_n)} \xmapsto[\text{linear map.}]{h} \Delta_{(hg_0,\ldots,hg_n)}$

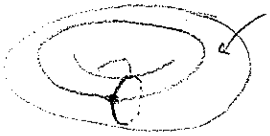
Set $BG = EG/G$ a Δ-complex.

Then
$$\begin{tikzcd} EG \arrow[d] \\ BG \end{tikzcd}$$
is a covering map and BG is a $K(G,1)$.

<u>Case $n > 1$</u>: 1) Start w/ $\bigvee_\alpha S^n_\alpha$ which have $\pi_i = 0 \quad i < n$
$\pi_n = \bigoplus_\alpha \mathbb{Z}$

First suppose there are finitely many spheres.

$$\underbrace{\bigvee_\alpha S^n_\alpha}_{n\text{-skeleton}} \longrightarrow \underbrace{\prod_\alpha S^n_\alpha}_{+\text{ cells of dim } 2n \text{ and up}}$$



$S^1 \vee S^1 = (T^2)^1$

$(\pi_n X) \cong (\pi_n X^{n+1})$

this map is an isom on π_n by cell approx, $\pi_n(\prod_\alpha S^n_\alpha) = \prod_\alpha \pi_n(S^n_\alpha)$.

In general get $\pi_n(\bigvee_\alpha S^n) \to \prod_\alpha \mathbb{Z}$ — inclusion of $S^n_\alpha \to \prod S^n_\alpha$

any $S^2 \to \bigvee_\alpha S^n_\alpha$ lies in finitely many S^n_α by cptness.

so image is actually $\bigoplus_\alpha \mathbb{Z}$.

---

2) $$0 \to K \to \bigoplus_\alpha \mathbb{Z} \to G \to 0$$

$$\bigoplus_\alpha \mathbb{Z} = \pi_n(\bigvee_\alpha S^n_\alpha)$$

and attach an $n+1$ cell to $\bigvee_\alpha S^n_\alpha$ for each $\varphi_\beta \in K$ via $\partial D^{n+1} \xrightarrow{\varphi_\beta} \bigvee_\alpha S^n_\alpha$ to get X

$$\begin{tikzcd}
\pi_{n+1}(X, \vee_\alpha S^n_\alpha) \arrow[r] \arrow[d, phantom, "\cong" description] & \pi_n(\bigvee_\alpha S^n_\alpha) \arrow[r] & \pi_n(X) \arrow[r, "\partial"] \arrow[d, phantom, "\shortparallel"] & \pi_{n-1}(X, \vee_\alpha S^n_\alpha) \arrow[d, phantom, "\shortparallel"] \\
\pi_{n+1}(X / \vee_\alpha S^n_\alpha) = \bigoplus_\beta \mathbb{Z} & & G & 0
\end{tikzcd}$$

pair n-connected

$\vee_\alpha S^n_\alpha$ $n-1$ connected

So now have X which is $n-1$ connected and $\pi_n = G$.

But $\pi_{n+1}(X)$ might be non-zero. If so attach a bunch of $n+2$ cells along a gen set for $\pi_{n+1}(X)$.

$\left(\pi_n X^n \text{ gen } \pi_n X, \ \pi_n X^{n+1} \cong \pi_n X \right)$

Repeat.

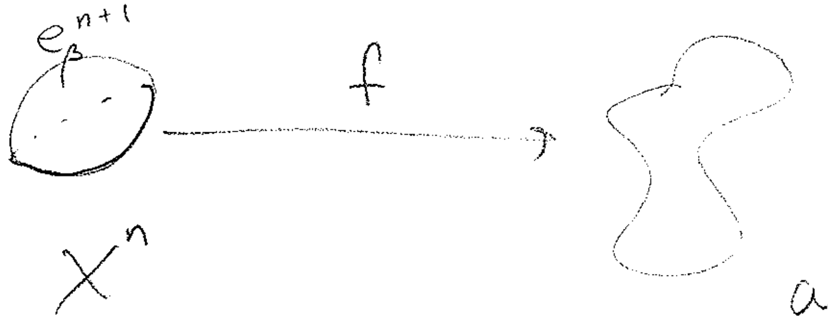
Pf of uniqueness. Enough to show that any $K(G,n)$ Y is $\simeq_{h.e.}$ to the X constructed above.

Let $\psi : \pi_n X \to \pi_n Y$ be an isom.

Send $X^n \xrightarrow{f} Y$ via $S^n_\alpha \xmapsto{\psi([S^n_\alpha])} Y$. f_* "implements" ψ

Extends over $n+1$ cells e^{n+1}_β by noting that

$f_*(\partial e^{n+1}_\beta) = \psi(\partial e^{n+1}_\beta) = 0$ hence $f|_{\partial e^{n+1}_\beta}$ extends over e^{n+1}_β.



Repeat. Now apply whiteheads thm.

Lecture 21: Last time: Eilenberg MacLane Spaces

[Copy thm, do proof of uniqueness.]

Note: For any X, $H^n(X,G) \cong$ hom classes of maps $X \longrightarrow K(G,n)$

[will do this.] $H^1(X,\mathbb{Z}) \cong [X,S^1]$ $H^2(X,\mathbb{Z}) \cong [X,\mathbb{C}P^\infty]$

Hurewicz Theorem: If X is an $(n-1)$ connected CW complex, $n \geq 2$ then $\tilde{H}_i(X)=0$ for $i<n$ and $\pi_n(X) \approx H_n(X)$.

Note: For any space have homomorphisms

$$h : \pi_k X \to H_k(X) \quad \text{sending } f : S^k \longrightarrow X \text{ to } f_*([S^k]).$$

but often these don't tell you anything

Ex: $\mathbb{C}P^\infty$ has lots of homology, but little homotopy (as see on Mon)

S^n has little homology, but lots of homotopy.

H.T. particularly useful in computing π_2, by passing to universal cover. $\pi_2 X \cong \pi_2 \tilde{X} \cong H_2(\tilde{X})$

$\pi_1 = 1$

$$\begin{tikzcd} \tilde{X} \arrow[d] \\ X \end{tikzcd}$$

Pf of HT: By Cor 4.16 may assume X has a single 0 cell and no cells of dim $< n$. $\Rightarrow \tilde{H}_i(X)=0$ for $i<n$ using cellular homology. As π_n and H_n are determined by X^{n+1}, can assume that X has no cells of dim $> n+1$. So $X = \bigvee_\alpha S^n_\alpha \cup_\beta e^{n+1}_\beta$

What β α th coor

As with last time gen by $D^{n+1} \longrightarrow e^{n+1}_\beta$

$S^n \longrightarrow \bigvee_\alpha S^n_\alpha \longrightarrow S_\alpha$

$$\begin{tikzcd}
\pi_{n+1}(X, \bigvee_\alpha S^n_\alpha) \arrow[r, "\partial"] \arrow[d, phantom, sloped, "\cong"] & \pi_n(\bigvee_\alpha S^n_\alpha) \arrow[r] \arrow[d, equal] & \pi_n X \arrow[r] \arrow[ddd, "\cong\ h"] & \pi_n(X, \bigvee_\alpha S^n_\alpha) \arrow[d, equal] \\
\pi_n(\bigvee_\beta S^{n+1}_\beta) \arrow[d, equal] & \bigoplus_\alpha \mathbb{Z} \arrow[dd, "\cong\ h"] \arrow[ddl, phantom, "\circlearrowright"] \arrow[ddr, phantom, "\circlearrowleft"] & & 0 \\
\bigoplus_\beta \mathbb{Z} \arrow[d, "h \cong"'] & & & \\
H_{n+1}(X, \bigvee_\alpha S^n_\alpha) \arrow[r, "\partial"] \arrow[d, equal] & H_n(\bigvee_\alpha S^n_\alpha) \arrow[r] \arrow[d, equal] & H_n(X) \arrow[r] & H_n(X, \bigvee_\alpha S^n_\alpha) \arrow[d, equal] \\
\bigoplus_\beta \mathbb{Z} & \bigoplus_\alpha \mathbb{Z} & & 0
\end{tikzcd}$$

degree $e^{n+1}_\beta \longrightarrow \bigvee_\alpha S^n_\alpha \Big/ \bigvee_{\text{not}\,\alpha} S^n_\alpha$

$\|$

Cellular boundary formula $\partial e^{n+1}_\beta = \sum d_{\alpha\beta}\cdot S^n_\alpha$

Lecture 22: Last time: Hurewicz Theorem / Today: Fiber bundles.

[Sequences of spaces so nice that you do get a long exact seq in π_*]

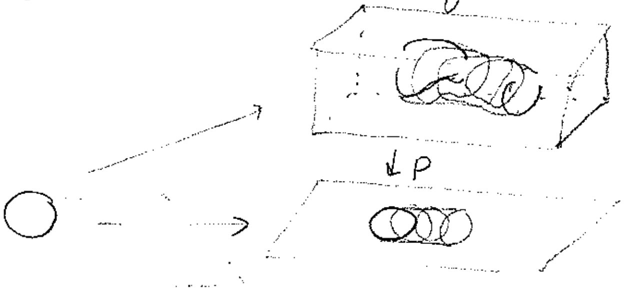
Homotopy lifting property: $p: E \to B$ has H.L.P. w.r.t X if given

$g_0 : X \to B$ and a lift

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, "\tilde{g}_0"] \arrow[r, "g_0"'] & B \end{tikzcd}$$

$p \circ \tilde{g}_0 = g_0$

every homotoy $g_t : X \times I \to B$ lifts to E starting at $\tilde{g}_0$.



Def: $p : E \to B$ is a fibration if it has H.L.P w.r.t. all X.

Ex: $E = B \times F$, p proj onto 1st factor. If $\tilde{g}_0 : X \to E$ is the initial lift $\tilde{g}_0(x) = (g_0(x), h(x))$ then take $\tilde{g}_t(x) = (g_t(x), h(x))$.

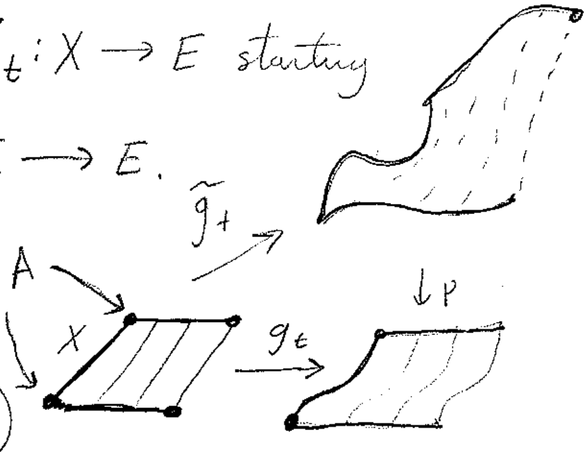
[Ex: $p : E \to B$ a covering space. ← Query]

[In general, $E \to B$ is a "twisted bundle w/ fiber F.]

Thm: Suppose $p: E \to B$ is a fibration. Let $b_0 \in B$ and $x_0 \in F = p^{-1}(b_0)$. Then if B is path conn, there is an exact sequence

$$\pi_n(F, x_0) \xrightarrow{i_*} \pi_n(E, x_0) \xrightarrow{p_*} \overset{\cong \pi_n(E,F,x_0)}{\pi_n(B, b_0)} \to \pi_{n-1}(F, x_0) \to \cdots$$
$$\cdots \to \pi_0(F, x_0) \to \pi_0(E, x_0) \to 0$$

Def: $p: E \to B$ has HLP for a pair (X, A) if each homotopy $g_t: X \times I \to B$ lifts to $\tilde{g}_t: X \to E$ starting w/ a given lift $\tilde{g}_0$ of g_0 and homotopy $A \times I \to E$.



Note: Any fibration has HLP w.r.t $(D^k, \partial D^k)$, as $(D^k \times I,\ D^k \times 0 \cup \partial D^k \times I) \cong (D^k \times I,\ D^k \times 0)$.

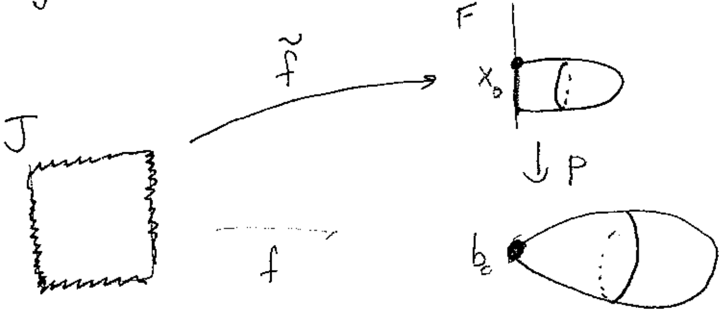
$\boxed{\text{Just explain above in the special case used in the theorem.}}$

Pf of Thm: Will show $p_*: \pi_n(E, F, x_0) \to \pi_n(B, b_0)$ is an isom. This gives the needed seq except for the very last 0; i.e. that every path comp of E contains part of F. But given $e \in E$ join $p(e)$ to b_0 by a path, then lifting to a path starting at e joins e to a pt in F.

p_* is onto: $f:(I^n,\partial I^n)\to(B,b_0)\in\pi_n(B,b_0)$

Let $J=I^n\setminus I^{n-1}$ and lift $f|_J$ by the const map to x_0.

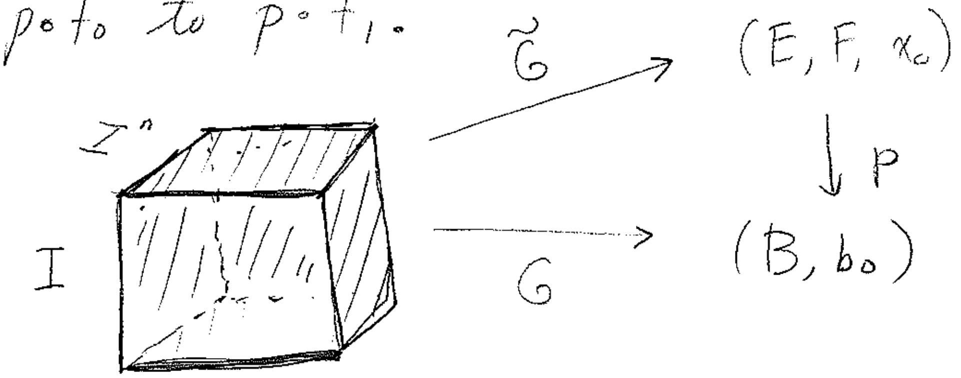
By HLP, $\tilde{f}|_J$ extends to a lift $\tilde{f}:I^n\to E$.



This is an elt of $\pi_n(E,F,x_0)$ as $\tilde{f}(\partial I^n)\subseteq F$ since $p\circ\tilde{f}(\partial I^n)=f(\partial I^n)=b_0$, which sat $p_*[\tilde{f}]=[p\circ\tilde{f}]=[f]$, as needed.

p_* is injective: Suppose $\tilde{f}_0,\tilde{f}_1:(I^n,\partial I^n,J)\to(E,F,x_0)$ rep elts of $\pi_n(E,F,x_0)$ s.t. $p_*([f_0])=p_*([f_1])$

Let $G:(I^n\times I,\partial I^n\times I)\to(B,b_0)$ be a homotopy from $p\circ f_0$ to $p\circ f_1$.



Have partial lift $\tilde{G}$ def on $\underset{(f_0,f_1)}{I^n\times\partial I}\cup\underset{\text{const map to } x_0}{J\times I}$

by HLP this extends, and the "front face" lands in F $\Rightarrow$ $\begin{matrix}[\tilde{f}_1].\\ \shortparallel\\ [\tilde{f}_0]\end{matrix}$

▨

Fiber Bundles: Locally a product.

$E \xrightarrow{p} B$ is a fiber bundle w/ fiber F if
"total space" "base space"

each pt in B has a nbhd U for which

there is a homeo

$$\begin{tikzcd}
p^{-1}(U) \arrow[rr, "h"] \arrow[dr, "p"'] \arrow[rr, phantom, "\circlearrowleft"', shift right=5] & & U \times F \arrow[dl, "\text{proj onto 1st fact}"] \\
& U &
\end{tikzcd}$$

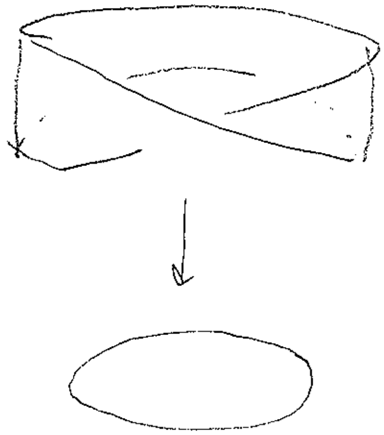
In particular

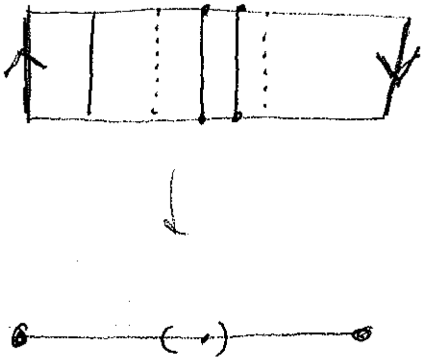
the fibers $p^{-1}(b)$ are all homeo to F.

Ex: 1) $E = B \times F$ 2) Covering space. (F a discrete set)

Ex: Möbius band

$I \to M \to S^1$





Next time: Hopf bundle. $S^1 \to S^3 \to S^2$

Lecture 23: Today: More of fiber bundles.

Last time: E [total space] $\downarrow p$ B [base] is a fiber bundle w/ fiber F if each pt in B has a nbhd U for which there is a homeo

$$\begin{tikzcd}
p^{-1}(U) \arrow[r, "h"] \arrow[dr, "p"'] & U \times F \arrow[d, "\pi"] \\
& U
\end{tikzcd}$$

$\circlearrowleft$ $\pi =$ proj onto 1st factor

Ex: $S^1 \to S^{2n+1} \to \mathbb{C}P^n$

all fibers are circles $\{\lambda(z_i) \mid \lambda \in S^1\}$

$$S^{2n+1} = \{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} \mid |z_0|^2 + \dots + |z_n|^2 = 1\}$$
$$P \downarrow$$
$$\mathbb{C}P^n = \mathbb{C}^n \setminus \{0\} / \mathbb{C}^*$$

If both in S^{2n+1}
$|\lambda| = 1 \quad \lambda \in S^1$
$(z_0, \dots, z_n) \longmapsto \lambda(z_0, \dots, z_n)$

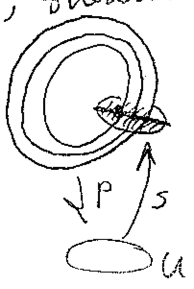
Local triviality: $U \subseteq \mathbb{C}P^n$ consist of $[z_0, \dots, z_n]$ with $z_0 \neq 0$.

$[U \cong \mathbb{C}^n$ via $[z_0, \dots, z_n] \longmapsto (z_1/z_0, \dots, z_n/z_0) \in \mathbb{C}^n]$

$h : p^{-1}(U) \to U \times S^1 \qquad h((z_0, \dots, z_n)) = ([z_0, \dots, z_n], z_0/|z_0|)$

$h^{-1}(\underbrace{[z_0, \dots, z_n]}_{\in S^{2n+1}}, \lambda) = \lambda |z_0| z_0^{-1} (z_0, \dots, z_n)$ well def for mult by $\alpha \in S^1$

Point: S^1 acts freely on S^{2n+1}, should give nice product structure.
Find a section $U \to S^{2n+1}$ intersecting each circle once will exhibit the product structure.



$S^1 \times s(U) \to p^{-1}(U)$
$\lambda \times (z_i) \longmapsto \lambda(z)$

$s(U) = \{(z_0, \dots, z_n) \in S^{2n+1} \mid z_0 \in \mathbb{R}\}$

$(z_0, \dots, z_n) \longmapsto \frac{|z_0|}{z_0}(z_i)$

Also works for $S^1 \to S^\infty \to \mathbb{C}P^\infty$.

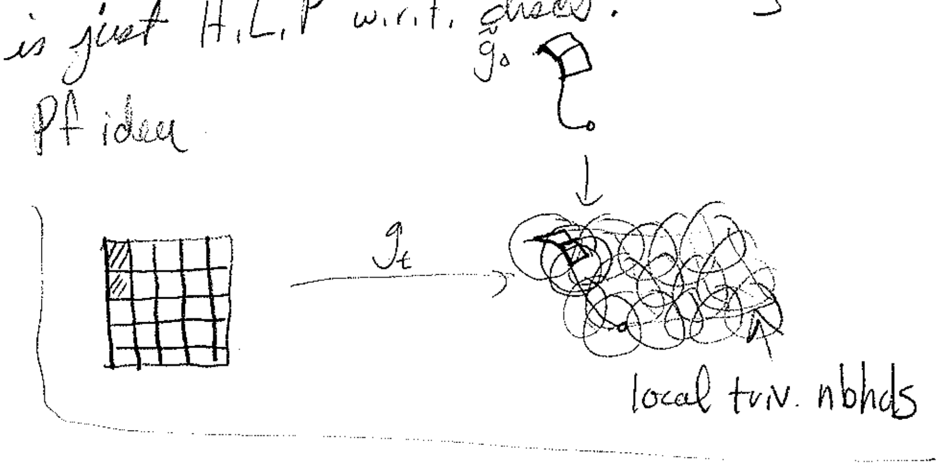
[Mention special case $n=1$ here]

Prop: A fiber bundle $p: E \to B$ has H.L.P. w.r.t. all CW spaces X.

[In fact, provided B is 2^{nd} countable, its really a fibration.]
What we need for L.E.S. is just H.L.P w.r.t. discs.

[Proof is simply local :]

Pf idea



Look at L.S.E. of this fibration

$$\pi_K S^1 \to \pi_K S^{2n+1} \xrightarrow{P_*} \pi_K \mathbb{C}P^n \to \pi_{K-1} S^1 \to \cdots$$

is an isom

If $K > 2$ ——

$\pi_K \mathbb{C}P^n = 0$ for $2 < K < n$ $\quad \pi_{2n+1} \underset{\text{2n-dimension}}{\underline{\underline{\mathbb{C}P^n}}} \cong \mathbb{Z}$.

For $K = 2$, know $\pi_2 \mathbb{C}P^n = \mathbb{Z}$ already.

For $\mathbb{C}P^\infty$ get that it is a $K(\mathbb{Z}, 1)$.

Special case: $n = 1$ $\quad \mathbb{C}P^1 \cong S^2$

$$S^1 \longrightarrow S^3 \longrightarrow S^2$$ Hopf bundle

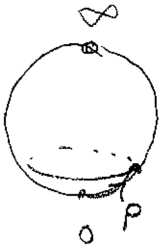
$$\underset{0}{\pi_3 S^1} \longrightarrow \underset{\mathbb{Z}}{\pi_3 S^3} \xrightarrow{\cong} \pi_3 S^2 \longrightarrow \underset{0}{\pi_2 S^1} \Rightarrow \pi_3 S^2 = \mathbb{Z}.$$

$$S^3 \subseteq \mathbb{C}^2 \xrightarrow{p} \mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$$
$$(z_0, z_1) \longmapsto z_0/z_1$$

$$p(r_0 e^{i\theta_0}, r_1 e^{i\theta_1}) = \underbrace{(r_0/r_1)}_{\rho} e^{i(\theta_0 - \theta_1)} \qquad r_0^2 + r_1^2 = 1$$

ρ fixed $\Rightarrow r_0, r_1$ fixed as well

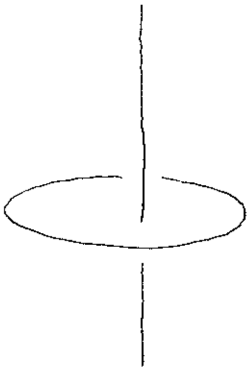
Thus T_ρ = set of pts in S^3 with fixed ρ



This is a torus where $\rho \in (0, \infty)$

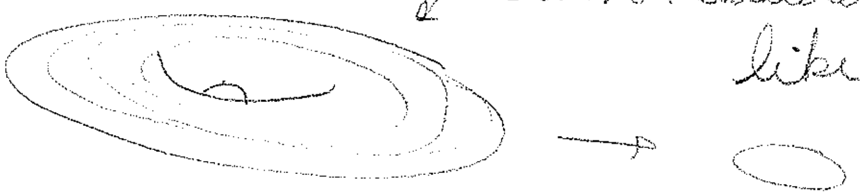
$T_0 = (0, e^{i\theta_1})$

$T_\infty = (e^{i\theta_0}, 0)$



Sterographically project

but not embedded like this



(over)

Note: S^2 and $S^3 \times \mathbb{C}P^\infty$

have isom π_n for each n.

[Query:] Not homotopy equiv as homology differs.

If time remains ~~⊘~~ ~~Pkze~~

$$\begin{tikzcd} & A \arrow[r, mapsto] & A\vec{e}_n \\ O(n-1) \arrow[r] & O(n) \arrow[r, "p"] & S^{n-1} \end{tikzcd}$$

fibration

$(n-1) + \dim O(n-1)$ $\frac{n^2}{2}$

$\pi_i O(n-1) \xrightarrow{\cong} \pi_i(O(n))$ for $i < n-2$

thus these ~~gps~~ π_i eventually stablize.

i mod 8	$\pi_i O$
0	$\mathbb{Z}/2$
1	$\mathbb{Z}/2$
2	0
3	$\mathbb{Z}$
4	0
5	0
6	0
7	$\mathbb{Z}$

Bott periodicity.

Lecture 24: Today: Stabilization.

Theme: large infinite-dim spaces can have simpler π_n than finite dimensional ones.

[finite number of cells ⇒ finite dim]

Rmk: There is no simply connected finite CW complex all of whose π_n are known, other than contractible ones.

S^n vs. S^∞, $\mathbb{CP}^n$ vs. $\mathbb{CP}^\infty$

(Whenever you have an increasing union, the π_n stab, assuming n-skeleton is fixed after some pt)

Bott Periodicity: $O(n) = \{A \in GL_n\mathbb{R} \mid$ pres. dot product $AA^T = I\}$

those fixing last vector

$$O(1) \leq O(2) \leq O(3) \leq O(4) \leq \cdots \qquad \begin{pmatrix} A & \begin{matrix}0\\0\\0\end{matrix} \\ 0\,0\,0 & 1 \end{pmatrix}$$

$\mathbb{Z}/2 \qquad S^1 \times \mathbb{Z}/2 \qquad \mathbb{RP}^3 \amalg \mathbb{RP}^3 \ldots$

Always two components det are ± 1.

Fiber bundle

$$O(n-1) \longrightarrow O(n) \longrightarrow S^{n-1}$$

$\Rightarrow \pi_i O(n-1) \longrightarrow \pi_i O(n)$ is an isom for $i < n-2$

i mod 8	0	1	2	3	4	5	6	7
$\pi_i O$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

And then it repeats!

[similar for U, Sp]

Stable Homotopy Groups:

$$\begin{tikzcd} S^i \arrow[d, "f"'] & SS^i = S^{i+1} \arrow[d] \\ X & SX \end{tikzcd}$$

X an n-connected space then $\pi_i(X) \xrightarrow[S]{\cong} \pi_{i+1}(SX)$

for $i < 2n+1$. If particular SX is $n+1$ connected

Now suppose X is any space

$$\pi_i(X) \longrightarrow \pi_{i+1}(\overset{\text{path connec}}{SX}) \longrightarrow \pi_{i+2}(\overset{\text{1-connected}}{S^2X}) \longrightarrow \pi_{i+3}(\overset{\text{2-conn}}{S^3X}) \longrightarrow \cdots$$

$$\pi_{i+k}(\underbrace{S^kX}_{\boxed{k-1}\text{ connected}}) \xrightarrow{\ \cong\ } \pi_{i+k+1}(S^{k+1}X)$$

$\cong$ provided $i+k < 2k-1$

Thus eventually these are all isom. Denote the "limiting" group $\pi_i^s X$ "stable homotopy group"

Special case: $\pi_i^s S^0 \cong \pi_{i+n} S^n$ for $n > i+1$.

π_i^s — stable i-stem [Easier to compute than the hom. gps themselfs; occur in many other contexts.]

i	0	1	2	3	4	5	6	7	8	9	
π_i^s	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	6	$\mathbb{Z}/2$	$\mathbb{Z}/240$	$\mathbb{Z}/2 \times \mathbb{Z}/2$	$(\mathbb{Z}/2)^3$	$\mathbb{Z}/6$

$\mathbb{Z}/504$ 0 ....

id $\uparrow$ (to $\mathbb{Z}$)

$\underset{\eta}{S^3 \to S^2}$ $\uparrow$ (to $\mathbb{Z}/2$)

$\underset{\nu}{S^7 \to S^4}$ $\uparrow$ (to $\mathbb{Z}/24$)

$S^{15} \longrightarrow S^8$ $\uparrow$ (to $\mathbb{Z}/240$)

Always finite for $i > 0$.

[Not without structure]

Product: $\pi_i^s \times \pi_j^s \longrightarrow \pi_{i+j}^s$

$f \, , \, g \longmapsto f \circ g : S^{n+i} \to S^{n-j}$

$f : S^{n+i} \to S^n$

$g : S^n \to S^{n-j}$

Thm: The product above makes $\pi_*^s = \oplus_i \pi_i^s$ into a graded ring w/ com. relation $\alpha \cdot \beta = (-1)^{|\alpha||\beta|} \beta\alpha$.

Pf: See text.

Many products are 0, but this lets us construct some additional elements too.

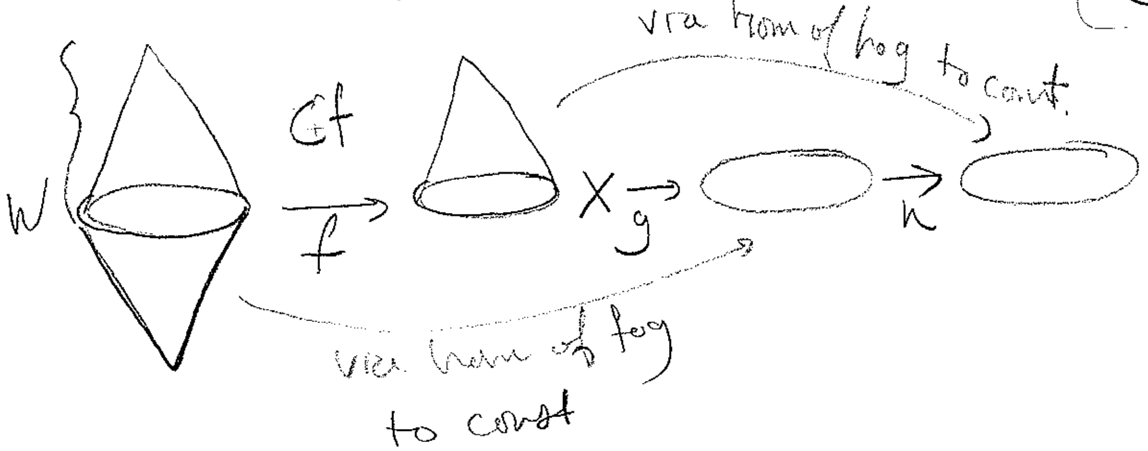
$$\begin{tikzcd}
 & \pi_i^s & \pi_j^s & \pi_k^s \\
W \arrow[r, "f"] & X \arrow[r, "g"] & Y \arrow[r, "h"] & Z \\
S^{i+j+k+l} & S^{j+k+l} & S^{k+l} & S^l
\end{tikzcd}$$

$f \circ g$, $\boxed{h \circ g}$ null homp.

$S^{i+j+k+l+1} \to S^l$

$\boxed{\pi^s_{i+j+k+1}}$

Toda Braket: $\langle f, g, h \rangle : SW \longrightarrow Z$



Discuss looking at the p-part, etc.

Lecture 25: Cohomology via $K(G,n)$s.

Def: X, Y spaces w/ basepts. Then $\langle X, Y \rangle =$ base-pt preserving hom classes of maps $X \to Y$

Ex: $\pi_n X = \langle S^n, X \rangle$

$[X, Y]$ hom. class of maps $f : X \to Y$

Thm: Let X be a CW complex, G an abelian gp, $n > 0$. Then $\exists$ a natural bijection $T : \langle X, K(G,n) \rangle \to H^n(X, G)$, which has the form $T([f]) = f^*(\alpha)$ for a certain fixed class $\alpha \in H^n(K(G,n); G)$.

Ex: $G = \mathbb{Z}$, $n = 1$ $\quad H^1(X; \mathbb{Z}) \cong \langle X, S^1 \rangle$

S^1 is a $K(G,n)$ $\quad f^*([S^1]^*) \longleftarrow\!\shortmid (X \xrightarrow{f} S^1)$

generator of $H^1(S^1)$, evaluates to 1 on $[S^1]$

One way to think about: Suppose X is connected.

Then $H^1(X; \mathbb{Z}) = \mathrm{Hom}(H_1(X); \mathbb{Z}) = \mathrm{Hom}(\pi_1 X; \mathbb{Z})$

Suppose $X \xrightarrow{f} S^1$. What hom is $f^*([S^1])$?

$\alpha \in \pi_1 X$. $\quad f^*([S^1]^*)(\alpha) = [S^1]^*(f_* \alpha) = f_*(\alpha)$ under the ident of $\pi_1 S^1 = \mathbb{Z}$

Thus
$$\begin{tikzcd} \langle X, S^1 \rangle \arrow[r] & \mathrm{Hom}(\pi_1 X, \mathbb{Z}) \\ f \arrow[r, mapsto] & f_*. \end{tikzcd}$$

Thm is equivalent to:

Any $\phi : \pi_1 X \to \mathbb{Z}$ can be realized by some $X \to S^1$.

Any two such realizations are homotopic.

Not hard to do geometrically. [Gives geometric meaning to cohomology.]

Rmk: $X \xrightarrow{f} Y$ then get precomposition

a map
$$\begin{tikzcd} \langle X, K(G,n) \rangle \arrow[d] & \langle Y, K(G,n) \rangle \arrow[l, "f^*"'] \arrow[d] \\ H^n(X,G) & H^n(Y,G) \arrow[l, "f^*"'] \end{tikzcd}$$

Hatcher says this, but I don't think its needed.

Rmk: [As long as X is connected] can replace

$\langle X, K(G,n) \rangle$ $\underrightarrow{\text{with}}$ $[X, K(G,n)]$

as obvious map is a bijection.

Approach to pf of Theorem: [More direct approaches possible.]

1) The functors $h^n(X) = \langle X, K(G,n)\rangle$ define a reduced cohomology theory on the category of CW spaces w/ basept. [Remarks above shows functorial nature.]

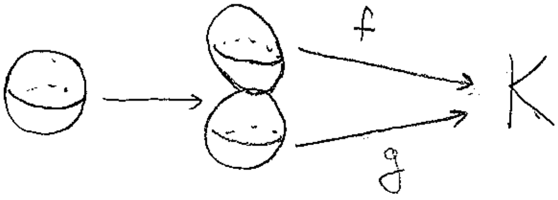
2) If a reduced cohom theory h^* has $h^n(S^0) = 0$ for all $n \neq 0$, then there are nat'l isom $h^n(X) \approx \widetilde{H}^n(X; h^0(S^0))$.

* * *

K_n a seq of spaces. When is $\langle X, K_n\rangle = h^n(X)$ a cohomology theory? [Mention Brown representability.]

[Have functoriality, Need group structure.]

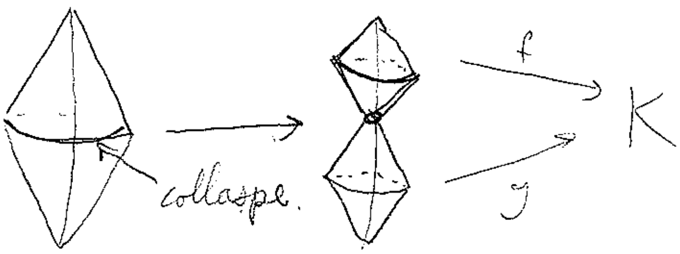
$\langle X, K\rangle$ $\quad X = S^n \quad$ $\langle S^n, K\rangle = \pi_n K$

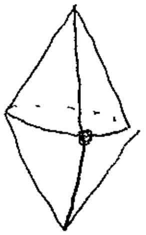


Base point issue so go with the reduced suspension:

$\Sigma X = SX/_{X_0 \times I}$

$SX \longrightarrow SX \vee SX$





If X_0 is a 0 cell $SX \to \Sigma X$ is a homotopy equiv

For any X, $\langle \Sigma X, K\rangle$ has a group structure.

Adjoint Relation:

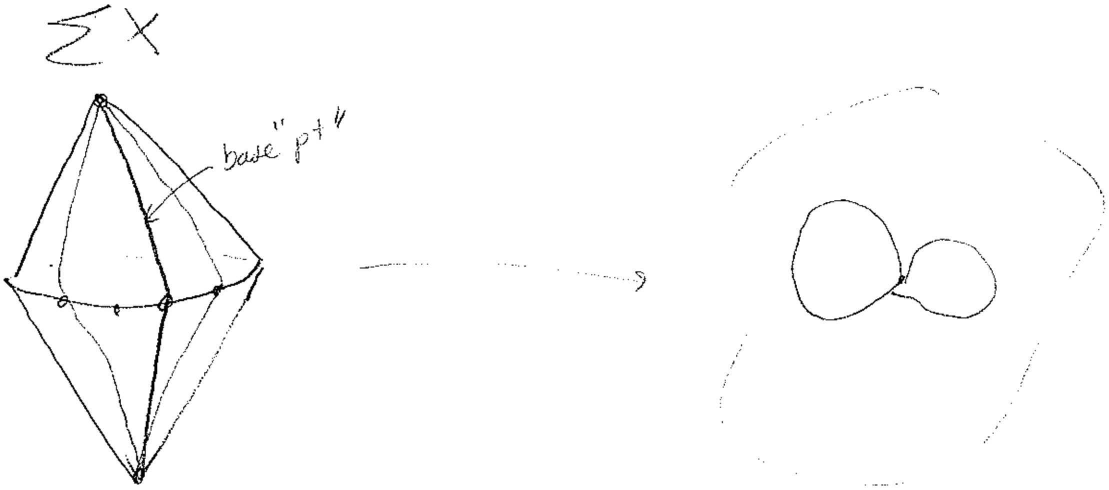
$$\langle \Sigma X, K\rangle \cong \langle X, \Omega K\rangle$$

where ΩK = loopspace of K = $\{$maps $I \to K$ taking both endpts to base pt $K_0\}$

$\subseteq K^I = \{f: I \to K\}$ with the compact-open topology.

basept: const loop at K_0.

Why:



$$\left(f: \Sigma X \longrightarrow K\right) \longmapsto \left(x \longmapsto f|_{x \times I}\right)$$

$$X \to \Omega K$$

Useful props: $\pi_{n+1}(K) = \pi_n(\Omega K)$

$$\pi_{n+1}(K) = \langle S^{n+1}, K \rangle = \langle \Sigma S^n, K \rangle$$

$$= \langle S^n, \Omega K \rangle = \pi_n(\Omega K)$$

So: $\Omega K(G,n)$ is a $K(G,n-1)$ | $\pi_n(K) = \pi_0(\Omega^n K)$

e.g. $\Omega \mathbb{C}P^\infty$ is homotopy equiv to S^1.

Lecture 26: Write up: Goal Thm from last time

$\Omega K = \langle S^1, K \rangle$ Adjoint relation. $\langle \Sigma X, K \rangle \cong \langle X, \Omega K \rangle$

Group structure on $\langle X, \Omega K \rangle$ [is induced from $\langle \Sigma X, K \rangle$]

$\Omega K \times \Omega K \longrightarrow \Omega K$ composition of loops.

$f, g \in \langle X, \Omega K \rangle \qquad (f+g)(x) = f(x) \cdot g(x)$

Not always abelian, but is for $\langle X, \Omega^2 K \rangle \approx \langle \Sigma^2 X, K \rangle$

$\Omega^2 K = \Omega(\Omega K) = \langle (I^2, \partial I), (K, k_0) \rangle$ Proof is the same as for π_2. $[\langle \Sigma^2 X, K \rangle]$

Ω-spectrum: A seq of CW complexes $K_1, K_2, \ldots$ together with homotopy equivalences $K_n \to \Omega K_{n+1}$

Ex: Fix an abelian gp G, take $K_n = K(G, n)$.

Ex: $O = \bigcup O(n)$ Then $O \simeq_{h.e.} \Omega^8 O$ (Bott period)

$K_n = \Omega^{(+n \bmod 8)} O$ $\qquad \leftarrow \Omega^2 O \quad \Omega O \leftarrow O \leftarrow \Omega^7 O$

$O \leftarrow \Omega^7 O$

Thm: If K_n is an Ω-spectrum, then the functors $X \longmapsto h^n(X) = \langle X, K_n \rangle$ for $n \in \mathbb{Z}$ define a reduced cohomology theory on the category of CW complexes with base pt.

[for O spectrum, get real K-theory.]

Axioms: $\underset{\text{a functor}}{h^n : CW \to Ab}$ + nat'l coboundary map $\delta : h^n(\overset{\text{subcplx of } X}{A}) \to h^{n+1}(X/A)$

1) homotopic maps induce same map on h^n

2) long exact seq of a pair

3) $h^n(\bigvee_\alpha X_\alpha) \cong \prod_\alpha h^n(X_\alpha)$ [Via the obvious isomorphisms

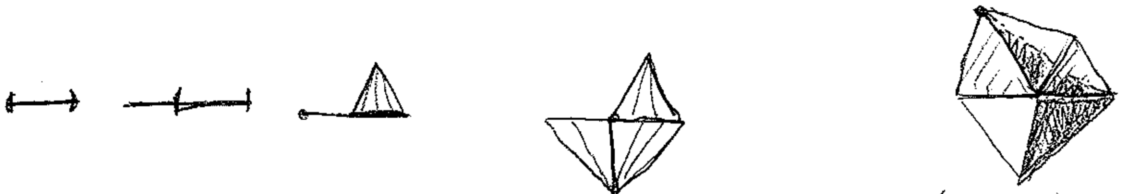
$$\begin{tikzcd} \bigvee_\alpha X_\alpha \arrow[r, "\pi_\alpha"] & X_\alpha \\ h^n(X) & h^n(X_\alpha) \arrow[l, dashed] \end{tikzcd}$$

]

Pf of thm. 1) $f : X \to Y$ get $\langle X, K_n \rangle \xleftarrow{f^*} \langle Y, K_n \rangle$ clearly dep only on homotopy class. f^* is a hom $K_n = \Omega K_{n+1}$

3) $\bigvee_\alpha X_\alpha \longrightarrow K_n \;=\; \{X_\alpha \to K_n\}$ provided all maps are base pt pres.

Need to build long exact sequence for X, A



adding cone of two steps previous

$\star\ A \hookrightarrow X \hookrightarrow X \cup CA \hookrightarrow (X \cup CA) \cup CX \hookrightarrow ((X \cup CA) \cup CX) \cup C(X \cup CA) \hookrightarrow$

$A \hookrightarrow X \hookrightarrow \overset{\|_{h.e.}}{X/A} \longrightarrow \underbrace{\overset{\|_{h.e.}}{\Sigma A} \hookrightarrow \overset{\|}{\Sigma X}}$

↑ [fuzzy]

— original pair suspended up.

$\longrightarrow \Sigma(X/A) \longrightarrow \Sigma^2 A \hookrightarrow \Sigma^2 X \longrightarrow \Sigma^2(X/A) \to$

Take hom classes of maps to K

[groups from here out, abelian gps. from $\Sigma^2 A$ out]

$\langle A, K \rangle \leftarrow \langle X, K \rangle \leftarrow \langle X/A, K \rangle \leftarrow \Big| \langle \Sigma A, K \rangle \leftarrow \langle \Sigma X, K \rangle \cdots$

↑ have dist. elts.

<u>Claim</u>: Above is exact.

To see this, observe in $\star$ every 3 terms are the same, so suffices to check here

↓

$\langle A, K \rangle \leftarrow \langle X, K \rangle \leftarrow \langle X \cup CA, K \rangle$

$f|_A : A \to K \leftarrow f : X \to K$

↑ if null hom, extends over

Take $K = K_n$ in our seq

Earlier terms are what we want.

$$\begin{tikzcd}
 & & h^n(A) & h^n(X) & h^n(X/A) \\
 & & \langle A, K_n \rangle \arrow[d, "\cong"] & \langle X, K_n \rangle \arrow[l] \arrow[d, "\cong"] & \langle X/A, K_n \rangle \arrow[l] \\
 & & \langle A, \Omega K_{n+1} \rangle \arrow[d] & \langle X, \Omega K_{n+1} \rangle \arrow[l] \arrow[d] & {} \arrow[l] \\
 {} & \langle X/A, K_{n+1} \rangle \arrow[l] & \langle \Sigma A, K_{n+1} \rangle \arrow[l] & \langle \Sigma X, K_{n+1} \rangle \arrow[l] & \\
 & h^{n+1}(X,A) & & &
\end{tikzcd}$$

giving needed long exact sequence.

[This is nat'l because (*) is.]

<u>Rmk:</u> Based vs. Unbased Cohomology.

$\tilde{h}$ a reduced based theory

$h^n = \tilde{h}^n(X_+ = X$ w/ disjoint base pt$)$

is an unreduced unbased theory.

Lecture 27:

Last time:

Ω-spectrum: Sequence of spaces $\{K_n\}_{n\in\mathbb{Z}}$ together with homotopy equivalences $K_n \longrightarrow \Omega K_{n+1}$

Thm: If K_n is an Ω-spectrum, then

$h^n(X) = \langle X, K_n\rangle$ is a cohomology theory of CW complexes (with basepts).

Remainder of proof of thm. By last time, [for a CW pair (X,A)] have an exact seq.

$$\begin{tikzcd}
 & & & & \langle A, \Omega K_n\rangle = \langle A, K_{n-1}\rangle, \arrow[d, equal] & & \\
 & \langle A, K_n\rangle \arrow[d, equal] & \langle X, K_n\rangle \arrow[l] & \langle X/A, K_n\rangle \arrow[l] & \langle \Sigma A, K_n\rangle \arrow[l] & \langle \Sigma X, K_n\rangle \arrow[l] & \\
 {} & h^n(A) \arrow[l] & h^n(X) \arrow[l, "i^*"'] & h^n(X/A) \arrow[l, "p^*"'] & h^{n-1}(A) \arrow[l, "\partial"'] & h^{n-1}(X) \arrow[l] & {} \arrow[l] \\
 & \langle \Sigma A, \Omega K_{n+1}\rangle & & & \text{by definition} \arrow[u] & &
\end{tikzcd}$$

This gives the needed exact sequence by extending leftward.

The resulting exact seq is natural. $(X,A) \xrightarrow{f} (Y,B)$.

▨

Remark: Based v.s. Unbased Cohomology

$\tilde{h}^*$ a reduced based theory. Then

$$h^n(X) = \tilde{h}^n(X_+ = X \text{ w/ disjoint base pt})$$

switch order

is an unreduced unbased theory.

Brown representability thm: Let h^n be a red. coh. theory of based CW complexes. Then h^n comes from some Ω-spectrum.

Thm: G an abelian gp. Then $\exists$ a natural bijection

$$T : \langle X, K(G,n) \rangle \longrightarrow H^n(X;G)$$

[doesn't dep on X, of course.]

where X is a CW complex. More precisely, $\exists\, \alpha \in H^n(K(G,n);G)$ s.t. $T([f]) = f^*(\alpha)$.

Pf: Consider the Ω-spectrum

$$K_n = \begin{cases} K(G,n) & n > 0 \\ \Omega K(G,1) \simeq_{\text{h.e.}} \text{discrete set of } G \text{ points} & n = 0 \\ pt & n < 0 \end{cases}$$

By 1st theorem, this is a cohomology theory.

$$h^n(S^0) = \langle S^0, K_n \rangle = \pi_0(K_n) = \begin{cases} 0 & n > 0 \\ G & n = 0 \\ 0 & n < 0. \end{cases}$$

$\underset{s_0}{\circ}$ $\underset{s_1}{\circ}$

[Note π_0 is not a group.]

By Hatcher 4.59, this implies $\exists$ a <u>natural</u> isom^{T} between $h^n(X)$ and $H^n(X,G)$ [Basically sim pf that sing = cellular homology]

<u>Existence of α</u>: Let $\alpha \in H^n(K(G,n); G)$ be the elt corresponding to $K(G,n) \xrightarrow{id} K(G,n)$, i.e. $\alpha = T([id])$

Then for $[f] \in \langle X, K(G,n) \rangle$ we have

$$T([f]) = T\big(f^*[id]\big) = f^*\big(T[id]\big) = f^*(\alpha).$$

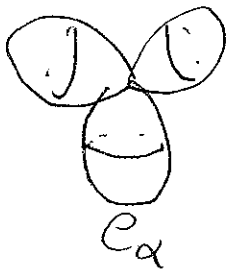
$$\begin{tikzcd}
\langle X, K(G,n) \rangle & \langle K(G,n), K(G,n) \rangle \arrow[l, "f^*"'] \\
{[f]} & {[id]} \arrow[l]
\end{tikzcd}$$

▨

Note can also specify α more geometrically.

Suppose K_n is the $K(G,n)$ constructed in class.

Then



α is the cellular-chain assigning to e_α the can. elt of $\pi_n K_n = G$.

Can also see cup product structure this way.

$$X \xrightarrow{f} K_n \qquad Y \xrightarrow{g} K_m$$

$$\begin{tikzcd}
X \times Y \arrow[r, "f\times g"] & K_n \times K_m \arrow[r] & K_n \wedge K_m \arrow[r, "\mu"] \arrow[d, equal] & K_{m+n} \\
 & n+m-1 \text{ connected} \arrow[ur] & K_n \times K_m / K_n \wedge K_m &
\end{tikzcd}$$

$$\begin{tikzcd}
\pi_{n+m} K_n \wedge K_m \cong & H_{n+m}(K_n \wedge K_m) \arrow[d] & = R \otimes R \arrow[d] \\
 & K_{m+n} & R
\end{tikzcd}.$$

If time remain, talk about bordism.