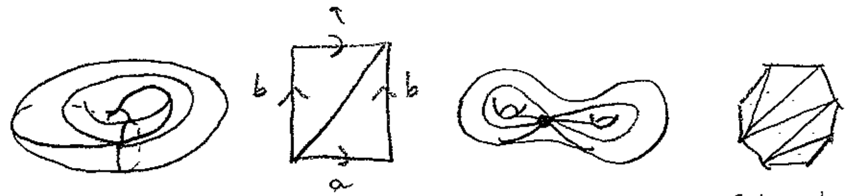
Lecture 23

Last time: $f: S^n \to S^n$

$\deg f = \#$ of pts in $f^{-1}(y)$ counted with sign $= \sum_{x \in f^{-1}(y)} \deg_x f$

Today: Cellular Homology.

Homology of a Δ-cplx:

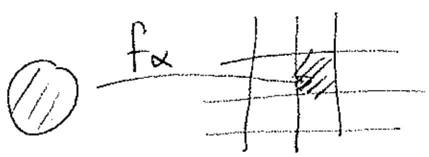


6 triangles

[would like to simplify.]

X a CW complex, $f_\alpha : D^{n_\alpha} \to X$, $f_\alpha : \partial D^{n_\alpha} \to X^{n_\alpha - 1}$

image a cell called e_α attaching map.



Fix a gen i_n of $H_n(D^n, D^{n-1})$.

$$H_K(X^n, X^{n-1}) \cong \tilde{H}_K\left(X^n/X^{n-1} \cong \bigvee_\alpha S^n\right) = \begin{cases} \bigoplus_\alpha \mathbb{Z} & n = K \\ 0 & \text{otherwise} \end{cases}$$

Define

$$C_K^{CW}(X) = H_K(X^K, X^{K-1}) = \bigoplus_\alpha \mathbb{Z} \quad \text{with basis } \{f_{\alpha *}[i_n]\}$$

$$\uparrow f_\alpha$$

$$H_K(D^K, D^{K-1})$$

[one gen for each n-cell just like simplicial hom]

Boundary map:

$$\begin{tikzcd}
 & H_K(X^K) \arrow[dr, "j"] & & & \\
H_{K+1}(X^{K+1}, X^K) \arrow[ur, "\partial"] \arrow[rr, "d_{K+1}"] & & H_K(X^K, X^{K-1}) \arrow[rr] \arrow[dr, "\partial"] & & H_{K-1}(X^{K-1}, X^{K-2}) \\
 & & & H_{K-1}(X^{K-1}) \arrow[ur, "j"] &
\end{tikzcd}$$

Boundary map for C_K^{CW}: $\boxed{d_K = j \circ \partial}$

Note: $d_K \circ d_{K+1} = 0$

as $\xrightarrow{j} \xrightarrow{\partial}$ is part of the long exact seq of the pair.

Def: $H_*^{CW}(X)$ is the hom. of $C_*^{CW}(X)$.

Thm: $H_*^{CW}(X) = H_*(X)$

Pf idea: Only If Asked

$$\begin{tikzcd}[column sep=small]
 & & H_k(X^k) \arrow[r, "i"] \arrow[d, "j"', "\cong"] & H_k(X^{k+1}) \arrow[r] & H_k(X) \\
 & H_{k+1}(X^{k+1}, X^k) \arrow[ur, "\partial"] \arrow[r, "d_{k+1}"] & H_k(X^k, X^{k-1}) \arrow[rr, "d_k"] \arrow[dr, "\partial"] & & H_{k-1}(X^{k-1}, X^{k-2}) \\
 & & & H_{k-1}(X^{k-1}) \arrow[ur, "j", "\cong"'] & \\
 & & H_{k-1}(X^{k-2}) \arrow[ur] & &
\end{tikzcd}$$

$j^{-1}(c) \mapsto c \in H_k(X^k, X^{k-1})$

$d_k(c) = 0 \Rightarrow \partial c = 0$

$[c] \in H_k^{CW}(X)$

$\downarrow$

$i(j^{-1}(c)) \in H_k(X^{k+1}) \cong H_k(X)$ ← by an easy ind. arg. if X is finite dim'l.

$H_{k-1}(X^{k-2}) = 0$ by HW from last week.

[Leave showing that it is an isom to the listener.]

Geometric Meaning of d_n: e_α an k-cell with map $f: D^k \to X$

$$d_k e_\alpha = \sum_{\beta \text{ } n-1 \text{ cells}} c_\beta e_\beta$$

$$\begin{tikzcd}
H_k(D^n, \partial D^{n-1}) \arrow[r, "f_*"] \arrow[d, "\partial"] & H_k(X^k, X^{k-1}) \arrow[r, "d_n"] \arrow[d, "\partial"] & H_{k-1}(X^{k-1}, X^{k-2}) \arrow[d, "q"] & \\
H_{k-1}(\partial D^{k-1}) \arrow[r, "f_*"] & H_{k-1}(X^{k-1}) \arrow[ur, "j"] & \tilde{H}_{k-1}(\bigvee_\beta S^{k-1}) \arrow[r] & \tilde{H}_{k-1}(S^{k-1}_\beta)
\end{tikzcd}$$

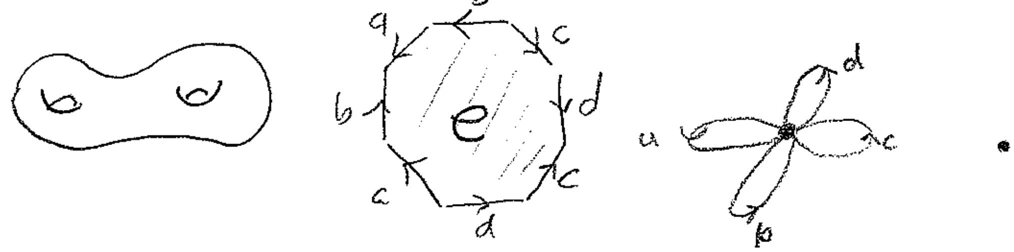
$i_k \longmapsto e_\alpha$

$H_{k-1}(\partial D^{k-1}) = \mathbb{Z}_{k-1}$

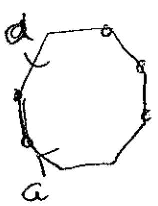
collapse all but β^{th} sphere.

$\sum c_\beta e_\beta \qquad \tilde{H}_{k-1}(S^{k-1}_\beta) = c_\beta \cdot \underbrace{\mathbb{Z}_{n-1}}_{\text{standard gen.}}$

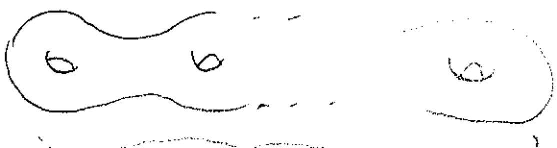
$c_\beta = \deg\left(f|_{\partial D} \to X^k/(X^{k-1} \cup \text{all other } k \text{ cells})\right)$

Ex: 

$$0 \longrightarrow C_2^{CW}(X) = \mathbb{Z} \xrightarrow[d_2=0]{} C_1^{CW}(X) = \mathbb{Z}^4 \xrightarrow[d_1=0]{} C_0^{CW}(X) = \mathbb{Z} \to 0$$

$d_2(e)$ along a  $= \deg\left(f_e : S^1 \to X^1/_{b,c,d} = \right.$  $\left.\right)$

$= 0$



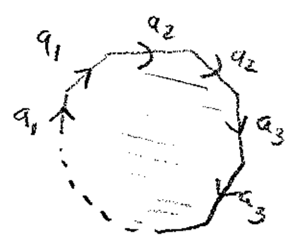
g holes

$= \Sigma_g \quad H_k(\Sigma_g) = \begin{cases} \mathbb{Z} & k=2 \\ \mathbb{Z}^{2g} & k=1 \\ \mathbb{Z} & k=0 \\ 0 & \text{otherwise} \end{cases}$

Ex: $\mathbb{R}P^2$  CW chains

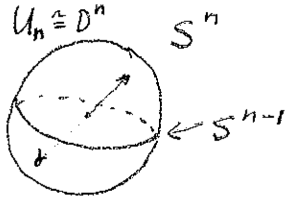
$$0 \longrightarrow \overset{2}{\underset{e}{\mathbb{Z}}} \xrightarrow{d_2} \overset{1}{\underset{a}{\mathbb{Z}}} \xrightarrow[0]{d_1} \overset{0}{\underset{v}{\mathbb{Z}}} \to 0$$

$\tilde{H}_1(\mathbb{R}P^2) = \mathbb{Z}/2$ all others 0. $\qquad d_2 : e \mapsto 2a$ as f_e is $z \mapsto z^2$

$\underbrace{\mathbb{R}P^2 \# \quad \# \mathbb{R}P^2}_{g}$  $\tilde{H}_1 = \mathbb{Z}/2 \oplus \mathbb{Z}^{g-1}$ all others 0.

$\mathbb{Z} \longrightarrow \mathbb{Z}^g$

$1 \longmapsto (2, 2, \dots, 2)$

Ex: $\mathbb{R}P^n \xleftarrow{q} S^n$ 

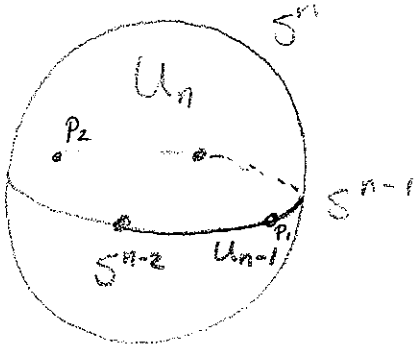
ident. antipodal points

$\mathbb{R}P^n = \mathbb{R}P^{n-1} \cup q(U_n)$

gives a cell str with one cell in each dim.

Computing the boundary maps

$$q : \partial U_n \longrightarrow X^{n-1}/X^{n-2} = \mathbb{R}P^{n-1}/\mathbb{R}P^{n-2} \cong \overset{p}{\overset{\cap}{U_{n-1}}}/\partial U_{n-1} \cong S^{n-1}$$



Pick $p \in \operatorname{int}(q(U_{n-1}))$. Then

$q^{-1}(p) =$ has two points p_1 and p_2

$$\deg_{p_1} q = 1$$

$$\deg_{p_2} q = \deg A = (-1)^n$$

So: $d_n = \begin{cases} \times 2 & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\times 2} \cdots \xrightarrow{0} \mathbb{Z} \to 0$$

$$0 \quad \mathbb{Z} \xrightarrow[0]{} \mathbb{Z} \xrightarrow[\times 2]{} \mathbb{Z} \xrightarrow[0]{} \mathbb{Z} \xrightarrow[\times 2]{} \cdots \to \mathbb{Z} \to 0$$

$$\tilde{H}_k(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & \text{for } k = n \text{ and } n \text{ odd} \\ \mathbb{Z}/2 & \text{for } k \text{ odd and } 0 < k < n \\ 0 & \text{otherwise.} \end{cases}$$

[Talk about $\mathbb{R}P^\infty$.]

This lecture was 5 min too long. Need to cut something for next time.

$\boxed{\text{Lecture 24}}$ Pickup HW. Ben McReynolds filling in on Mon.

Last time: Cellular homology $H_*^{CW}(X)$

Today: Euler char, Van Kampen for homology

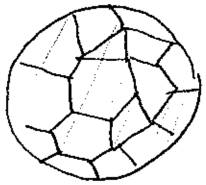
[Go back to finish up $\mathbb{R}P^n$]

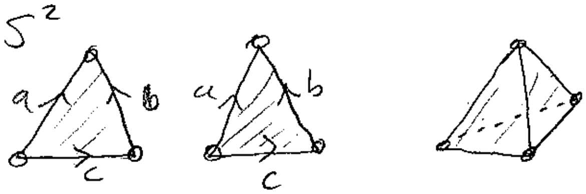
$\mathbb{R}P^n$ — CW complex with one cell U_n in each dim.

$$d_n(U_n) = \begin{cases} 2U_{n-1} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases} \qquad \tilde{H}_k(\mathbb{R}P^n) = \begin{cases} \mathbb{Z}/2 & 0<k<n \text{ and } k \text{ odd} \\ \mathbb{Z} & k=n \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$

n even $\quad 0 \longrightarrow \overset{n}{\mathbb{Z}} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{0} \cdots \xrightarrow{0} \mathbb{Z}$

n odd $\quad 0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \cdots\cdots \qquad \mathbb{Z}$

Euler's Thm:  S^2 divided into polygons. Then $(\#\text{ of vert}) - (\#\text{ edges}) + (\#\text{ faces}) = 2$.

Ex: S^2 

$4-6+4=2.$

$3-3+2=2$

[Easy Application: Show there are only five regular solids
tetrahedron, cube, octohedron, dodecahedron, icosohedron.]

Euler Characteristic: X a finite CW complex. Let $\chi(X) = \sum (-1)^i (\#\text{ of } i\text{-cells})$.

Thm: $X \simeq_{h.e.} Y$ finite CW complexes. Then $\chi(X) = \chi(Y)$.

[follows from:]

Thm: $\chi(X) = \sum(-1)^i \operatorname{rank}(H_i(X)) = \chi(H_*(X))$

means rank of free part $H_i(X) = \mathbb{Z}^{r} \oplus T$ finite.

Alg Lemma: $0 \to C_k \xrightarrow{\partial} C_{k-1} \xrightarrow{\partial} \cdots \to C_1 \to C_0 \to 0$

a chain complex with C_k finitely gen, H_* the result. homology

Set $\chi(C_*) = \sum(-1)^i \operatorname{rank} C_i$

Then $\chi(C_*) = \chi(H_*)$

Note: This implies Thm using CW homology

$$C_k^{CW}(X) = \begin{matrix}\text{free abelian}\\ \text{gp on cells}\end{matrix} \quad \text{so } \chi(C_*^{CW}(X)) = \chi(X).$$

Pf of Lemma: Fact: $0 \to A \to B \to C \to 0$
exact seq of finitely gen abelian groups.
Then $\operatorname{rank} B = \operatorname{rank} A + \operatorname{rank} C$. [Pf: Omit].

$Z_n = \ker \partial_n$, $B_n = \operatorname{im} \partial_{n+1}$, $0 \to B_n \to Z_n \to H_n \to 0$

$\operatorname{rank} Z_n = \operatorname{rank} H_n + \operatorname{rank} B_n$

Also $0 \to Z_n \to C_n \to B_{n-1} \to 0 \Rightarrow \operatorname{rank} C_n = \operatorname{rank} Z_n + \operatorname{rank} B_{n-1}$.

$\Rightarrow \operatorname{rank} C_n = \operatorname{rank} H_n + \operatorname{rank} B_n + \operatorname{rank} B_{n-1}$.

Sum times $(-1)^n$ proves the alg. lemma. ∎

Ex: $\chi($  $= \Sigma_g) = 1 - 2g + 1 = 2 - 2g$

$\chi(\#^n \mathbb{R}P^2) = 1 - n + 1 = 2 - n$

Ex: $\chi(S^n) = \begin{cases} 2 & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$ $\quad \chi(\mathbb{R}P^n) = \begin{cases} 1 & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$

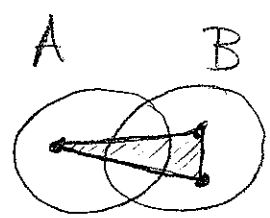
Note: $\chi(\mathbb{R}P^n) = 2\chi(S^n)$ $\quad$ Thm: $Y \to X$ a covering space of degree d. If X is a CW complex, then $d\chi(X) = \chi(Y)$.

Cor:  does not cover 

Gives another reason that only $\mathbb{Z}/2$ can act on even dim'l spheres (mod technical issues).

[Fact: Any odd dim'l manifold has $\chi = 0$]

Meyer-Vietoris: $A, B \subseteq X$ s.t. $X = \mathrm{Int}(A) \cup \mathrm{Int}(B)$



There is an exact seq

$$\to H_n(A\cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \to H_{n-1}(A \cap B) \to$$

Because of the short exact seq of chain complexes.

$$0 \to C_n(A\cap B) \to C_n(A) \oplus C_n(B) \to C_n^{\mathcal{U}}(X) \to 0$$

$$(x, y) \longmapsto x + y$$

$$z \longmapsto (z, -z)$$

$\mathcal{U} = \{A, B\}$ as in excision.

Homology w/ Coeff: G abelian gp, e.g. $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}/2$

$$C_n(X;G) = C_n(X) \otimes G = \bigoplus_{\sigma} G$$

σ sing n-simplex

Have same ∂ map, etc. Call Homology $H_n(X;G)$

has all usual prop, CW version.

$$\tilde{H}_k(S^n;G) = \begin{cases} G & n=k \\ 0 & \text{otherwise} \end{cases} \qquad \tilde{H}_k(D^{n+1};G) \to \cdots$$

Lecture 25

Today: Homology with coefficients

G = abelian group: $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{Z}/2$, etc.

means I wouldn't write this if I were lecturing.

$$C_n(X;G) = C_n(X) \otimes G = \bigoplus_{\substack{\text{sing. simp} \\ \sigma}} G$$ [sums of sing. simp w/ G-weights]

$$\partial\sigma = \sum (-1)^i \sigma|_{i\text{th face}}$$ [note: makes sense even when G is not a ring. $\partial^2 = 0$, etc.]

$H_n(X;G)$ = homology w/ G-coeff. [has all prop of hom we've seen so far, long exact seq, excision, cellular version, etc.]

Ex: $$\tilde{H}_k(S^n;G) \cong \begin{cases} G & n=k \\ 0 & \text{otherwise} \end{cases}$$

[Exactly two sing. simp. in each dim]

1) For $S^0 = \bullet \quad \bullet$ Have $C_k(S^0;G) = G \oplus G$

and !

$$\xrightarrow{0} G\oplus G \xrightarrow{\cong} \overset{C_1}{G\oplus G} \xrightarrow{0} \overset{C_0}{G\oplus G} \xrightarrow{\varepsilon} G$$

augmentation

so agrees with above and $\tilde{H}_0(S^0;G) = \ker \varepsilon = (g,-g) \subseteq G\oplus G$.
$\cong G$

2) In gen compute inductively as in the usual case

$$\longrightarrow \underset{\substack{\parallel \\ 0}}{\tilde{H}_k(D^{n+1};G)} \to H_k(D^{n+1},\partial D^{n+1};G) \to \tilde{H}_k(\underset{\substack{\sim \\ S^n}}{\partial D^{n+1}};G) \to \underset{\substack{\parallel \\ 0}}{\tilde{H}_k(D^{n+1};G)}$$

$$H_k(D^{n+1},\partial D^{n+1};G) \xrightarrow{\cong} H_k(D^{n+1}/\partial D^{n+1} \cong S^{n+1};G)$$

[To compute more examples, we want to introduce cellular hom w/ G-coeff. For this, we 1st must understand degree of $f: S^n\circlearrowleft$ with G coeff.]

<u>Lemma</u>: $f: S^n \circlearrowleft$ of deg d. Then $f_*: \tilde{H}_n(S^n;G) \circlearrowleft$ is $\underbrace{\text{mult by } d.}$

[really $x \mapsto \underbrace{x+\cdots+x}_{d \text{ times}}$]

<u>Gen fact</u>: $\varphi: H \to G$ hom of abelian gp.

get $\varphi_*: H_*(X;H) \to H_*(X;G)$ a hom. of abelian gps.

<u>Pf</u>: $c \in \tilde{H}_n(S^n;G)$. Let g be the corr. elt of G under the isom $\tilde{H}_n(S^n;G) \cong G$. Set $\varphi: \mathbb{Z} \to G$ via $1 \mapsto g$. Then

$\varphi_*\left(\begin{array}{c}\text{gen of } \tilde{H}_n(S^n;\mathbb{Z}) \\ \text{to } 1 \text{, call it } \alpha\end{array}\right) = g$ by induct using the formula above. Then

$$\begin{tikzcd}
\alpha \arrow[rr, mapsto] & & (\deg f)\alpha \\
\mathbb{Z} \cong H_n(S^n;\mathbb{Z}) \arrow[rr, "f_*"] \arrow[d, "\varphi_*"] & \circlearrowleft & H_n(S^n;\mathbb{Z}) \cong \mathbb{Z} \arrow[d, "\varphi_*"] \\
G \cong \tilde{H}_n(S^n;G) \arrow[rr, "f_*"] & & \tilde{H}_n(S^n;G) \cong G \\
c \arrow[rr, mapsto] & & (\deg f)c
\end{tikzcd}$$

as required. ∎

[↗ The pt is that once we fix our ident of $D^{n+1}/\partial D^{n+1} \cong S^{n+1}$ the above gives a <u>canonical</u> isom between $\tilde{H}_n(S^n;G) \cong G$ comming from $\ker \varepsilon \cong G$]

<u>CW homology w/ coeff</u>: $C_n^{CW}(X) = H_n(X^n, X^{n-1}; G) = \bigoplus_{n\text{-cells}} G$,

$d_n = j \circ \partial_n$ as before.

<u>Lemma says</u>: can compute d_n using deg. of maps of spheres, as before.

$$d_n e_\alpha = \sum_{\substack{n-1 \text{ cells} \\ e_\beta}} c_{\alpha\beta} \cdot e_\beta$$

again stress notation

$c_{\alpha\beta}$ = degree of attaching map of e_α to image of e_β in $X^{n-1}/(X^{n-1} \setminus \mathring{e}_\beta)$.

Ex: $\mathbb{R}P^n$ [one cell in every dim]

$G = (\mathbb{R}, +)$.

C^{CW}

[really mean $a+a$ — we don't always have a ring structure.]

$$\xrightarrow{\times 2} \overset{3}{\mathbb{R}} \xrightarrow[0]{} \overset{2}{\mathbb{R}} \xrightarrow[\times 2]{} \overset{1}{\mathbb{R}} \xrightarrow[0]{} \overset{0}{\mathbb{R}} \to 0$$

So $\tilde{H}_k(\mathbb{R}P^n; \mathbb{R}) = \begin{cases} \mathbb{R} & n = k \text{ and } k \text{ is odd.} \\ 0 & k \neq n \end{cases}$

$G = (\mathbb{Z}/2, +)$ — chains as collections of cells, w/o orient.

$$\cdots \xrightarrow{0} \mathbb{Z}/_2 \xrightarrow{\times 0} \mathbb{Z}/_2 \xrightarrow{\times 2 = 0} \mathbb{Z}/_2 \xrightarrow{0} \mathbb{Z}/2 \to 0$$

$\tilde{H}_k(\mathbb{R}P^n; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & 0 < k \leq n \\ 0 & \text{otherwise.} \end{cases}$

[for a gen field, just the fn of its characteristic.]

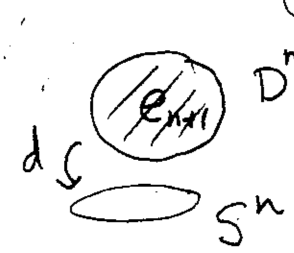
Fact: No new info is created by using gen coeff (§3.A)

Still usefull, though, esp field coeffs. [building spaces one prime at a time.]

Exercise:
X a finite CW complex. F a field. Then

$$\chi(X) = \sum (-1)^i \dim H_i(X; F).$$

<u>Moore Space</u>: $X = S^n$ w/ one $n+1$ cell attached by a map of deg d.



Consider

$$X \xrightarrow{q} X/S^n = S^{n+1}$$

<u>Q</u>: is q null homotopic? [what is action on homology]

	$n+1$	n	0	0
$\mathbb{Z}$-homology of X	$\mathbb{Z} \xrightarrow{\times d}$	$\mathbb{Z} \longrightarrow$	0	$\mathbb{Z}$
$\tilde{H}_*(X)$	0	$\mathbb{Z}/n$		0

so $q_* = 0$ on homology. However, for

$\mathbb{Z}/d$ homology:

$H_{n+1}(X; \mathbb{Z}/d) \overset{!!}{\cong} \mathbb{Z}/d$. and $q_* : H_{n+1}(X, \mathbb{Z}/d) \xrightarrow{\cong} H_{n+1}(S^{n+1}; \mathbb{Z}/d)$.

Thus q is not null homotopic.

[So other coeff can be usefull...]

<u>Next time</u>: The formal viewpt (categories, functors, etc). Hatcher §2.3

Lecture 26 Last time: Homology w/ coeffs

[How many diff notions of homology are there?]

Today: The formal viewpt (Section 2.3).

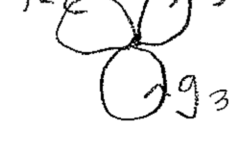
Category: $\mathcal{C}$ consists of

1) A coll of objects $Ob(\mathcal{C})$
2) $\forall X, Y \in Ob(\mathcal{C})$ have $Mor(X,Y)$ the "morphisms."
   Have special $1_X \in Mor(X,X)$ the "identity".
3) Composition of morphisms $\circ : Mor(X,Y) \times Mor(Y,Z) \to Mor(X,Z)$
   $\forall X, Y, Z \in Ob(\mathcal{C})$ sat. $f \circ (g \circ h) = (f \circ g) \circ h$ and $f \circ 1 = 1 \circ f = f$.

Ex: 1) Category of top. spaces $Ob =$ all top spaces, $Mor(X,Y) =$ cont maps.

2) Category of Abelian Groups: $Mor(X,Y) =$ homs from $X \to Y$.

3) G a group. $Ob = p$. $Mor(p,p) =$ elts of G



$\circ =$ group mult.

[sat 2) and 3) because is a gp]

[A cat w/ one obj is a "group w/o inverses," aka a monoid.]

A **Functor** from $\mathcal{C}$ to $\mathcal{D}$ is $X \in Ob(\mathcal{C}) \to F(X) \in Ob(\mathcal{D})$

$f \in Mor(X,Y) \longrightarrow F(f) \in Mor(F(X), F(Y))$.

$X, Y \in Ob(\mathcal{C})$

sat $F(1_X) = 1_{F(X)}$

$F(f \circ g) = F(f) \circ F(g)$

Ex: 1) Singular homology $H_n : \begin{pmatrix} \text{cat of} \\ \text{top spaces} \end{pmatrix} \to \begin{pmatrix} \text{cat of} \\ \text{abelian gps} \end{pmatrix}$

$X \qquad\qquad H_n(X; \mathbb{Z})$

2) Abelianization: $\begin{pmatrix} \text{cat of} \\ \text{groups} \end{pmatrix} \longrightarrow \begin{pmatrix} \text{cat of} \\ \text{abelian gps} \end{pmatrix}$

Q: do π_1 a functor? A: yes but only from the category of top spaces w/ basepoints.

[N.B.: These are covariant functors. There are also contravariant ones [e.g. cohomology]]

Natural Transformations: $F, G : \mathcal{C} \to \mathcal{D}$ functors.

$\forall X \in Ob(\mathcal{C}) \quad T_X : F(X) \to G(X)$ a

such that $\forall f \in Mor(X,Y)$ then
$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "T_X"'] \arrow[dr, phantom, "\circlearrowright"] & F(Y) \arrow[d, "T_Y"] \\ G(X) \arrow[r, "G(f)"'] & G(Y) \end{tikzcd}$$
commutes.

Ex: $\varphi : G_1 \to G_2$ hom of abelian gps.

$$F, G : \mathcal{J}_{op} \longrightarrow Abel. \quad F = H_n(X; G_1)$$
$$G = H_n(X; G_2)$$

$T_X : F(X) \xrightarrow{\varphi_*} G(X)$ from last time.

used nat'l. prop of T_X on Monday for deg of maps of spheres.

A reduced homology theory is:

A functor $\tilde{h} : \begin{pmatrix} \text{CW complexes} \\ + \text{cont maps} \end{pmatrix} \longrightarrow \begin{pmatrix} \text{seq of abelian} \\ \text{gps} \end{pmatrix}$

$$X \longrightarrow \{\tilde{h}_n(X)\}_{n \in \mathbb{Z}}$$

sat:

1) $f \simeq g : X \to Y$ then $\tilde{h}(f) = \tilde{h}(g)$

2) $A \subseteq X$ a subcomplex. Consider the two functors.

$$\begin{pmatrix} \text{CW complex w} \\ \text{a sub comp} \end{pmatrix} \longrightarrow \begin{pmatrix} \text{seq of} \\ \text{abelian groups} \end{pmatrix}$$

$$(X, A) \longrightarrow \{\tilde{h}_n(X/A)\}$$
$$\longrightarrow \{\tilde{h}_{n-1}(A)\}$$

Then $\exists$ a natural transformation ∂

$\partial : \tilde{h}_n(X/A) \to \tilde{h}_{n-1}(A)$ s.t.

$$\xrightarrow{\partial} \tilde{h}_n(A) \xrightarrow{\tilde{h}(i)} \tilde{h}_n(X) \to \tilde{h}_n(X/A) \xrightarrow{\partial} \tilde{h}_{n-1}(A) \to \cdots$$

is exact.

3) $X = \bigvee_\alpha X_\alpha$ then $\bigoplus_\alpha \tilde{h}_n(X_\alpha) \xrightarrow[\bigoplus_\alpha \tilde{h}(i_\alpha)]{\cong} \tilde{h}_n(X)$.

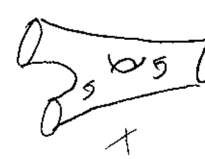
Ex: Fix some gp G, $\tilde{h}_n(X) = \tilde{H}_n(X;G)$

Ex: K-theory, bordism.

Thm: Suppose $\tilde{h}$ is a homotopy theory of CW complexes.

If $\tilde{h}_n(S^0) = \begin{cases} G & n=0 \\ 0 & \text{otherwise} \end{cases}$, then $\tilde{h}_n(X) = H_n(X;G)$.

Bordism: X space. $C_n^{\Omega}(X) = \begin{cases} f : M \to X \\ M \text{ smooth orient cpt } n\text{-mfld} \\ \text{poss w/ } \partial \end{cases}$

 $\to X$

opperations are disjoint union: $(+)$

change orient (mult by -1)

$\{\phi\}$ is zero. This is a free abelian gp.

$$C_n^{\Omega}(X) \xrightarrow{\partial} C_{n-1}^{\Omega}(X)$$

$$f : M \to X \longmapsto f|_{\partial M} : \partial M \to X.$$

$\partial^2 = 0. \Rightarrow H_n^{\Omega}(X)$ bordism homology.

Lecture 27

Last time: Not so relevant.

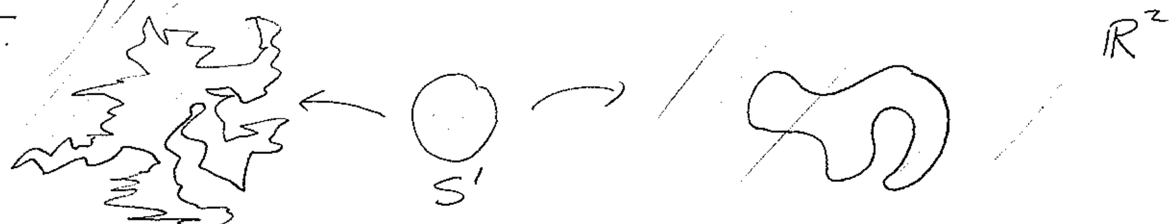
Reminder: Final Handed Out on Wed.

[Draw wild arc up ahead of time] 48

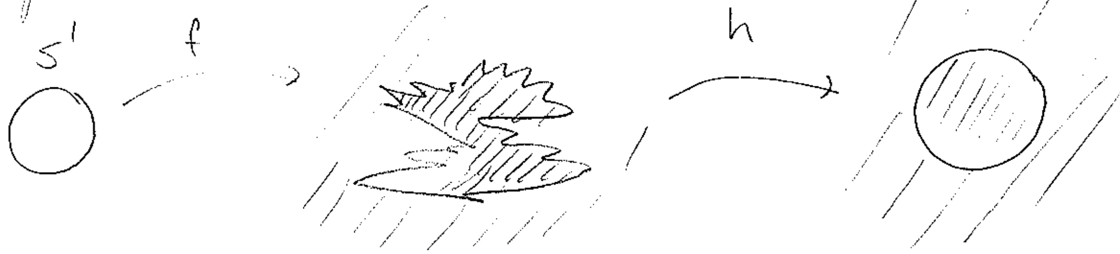
Today: Jordan Curve Thm, embeddings of S^k in $\mathbb{R}^n$

Jordan Curve Thm (1887): [But proof was wrong, 1st correct proof Veblen in 1905.]

$f: S^1 \hookrightarrow \mathbb{R}^2$ an embedding. Then $\mathbb{R}^2 \setminus f(S^1)$ has exactly two connected components.



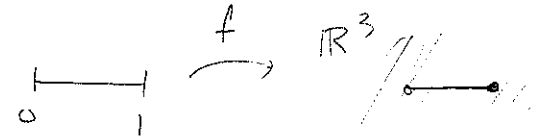
Schoenflies Thm (1906): $f: S^1 \hookrightarrow \mathbb{R}^2$. Then $\exists\, h: \mathbb{R}^2 \circlearrowleft \cong$ s.t. $h \circ f(S^1) = S^1$ and $h \circ f|_{S^1} = id$.



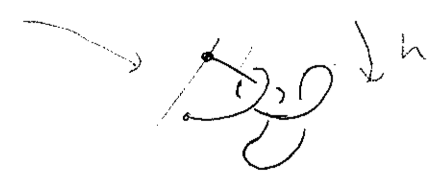
[Today and next time, will prove J.C.T. in all dims.]

Jordan Curve Thm generalizes to all $S^n \hookrightarrow \mathbb{R}^{n+1}$, but Schoenflies doesn't.

Wild arcs and spheres in $\mathbb{R}^3$: Standard:

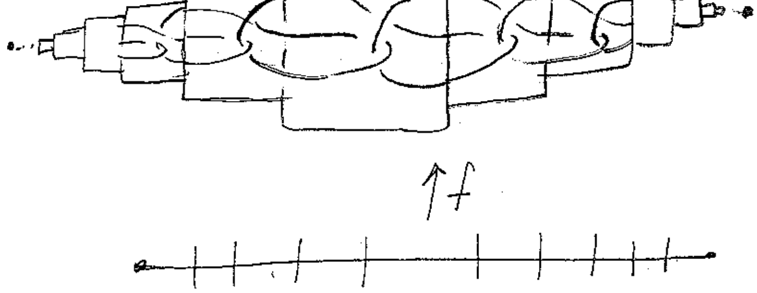


$\mathbb{R}^3 \setminus f(I) \simeq_{h.e.} S^2$



Wild:

each box is the same



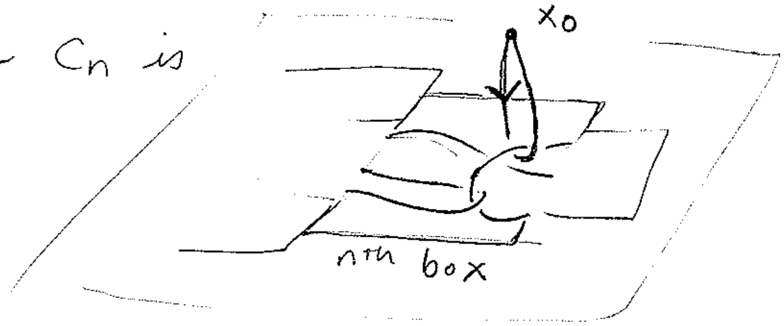
This gives a cont map $f: I \to \mathbb{R}^3$

[only pos. cont issue is at the ends

Using Van Kampen's Thm and a little geometric calculation

gives $\pi_1(\mathbb{R}^3 \setminus f(I)) = \left\langle \{C_n\}_{n\in\mathbb{Z}} \mid C_{n-1}C_nC_{n+1} = C_nC_{n+1}C_{n-1}C_n \right\rangle$

where C_n is



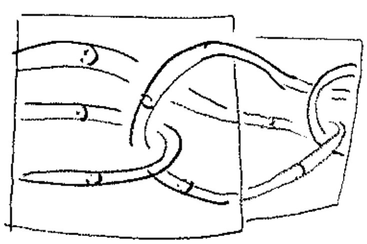
This group is non-trivial

$\rho : \pi_1 \to S_5$

$C_n \longmapsto \begin{cases} (12345) & n \text{ odd} \\ (14235) & n \text{ even} \end{cases}$

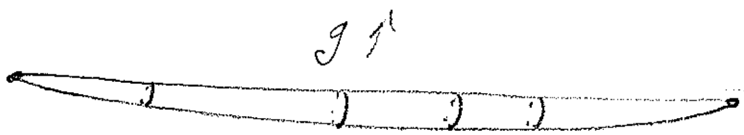
[So no homeo h of $\mathbb{R}^3$ taking this arc to the standard one.]

Thickening, gives a map $g: S^2 \hookrightarrow \mathbb{R}^3$ [where the S^2 gets ever thinner as you move toward the ends. So an embedding, homeo onto image]



Then $\pi_1(\mathbb{R}^3 \setminus g(S^2)) \neq 1$

so $\not\exists\, h: \mathbb{R}^3 \to \mathbb{R}^3$ taking $g(S^2)$ to the standard S^2.



[So Scoenflies fails in higher dims] [Also Alex. horned sphere

---

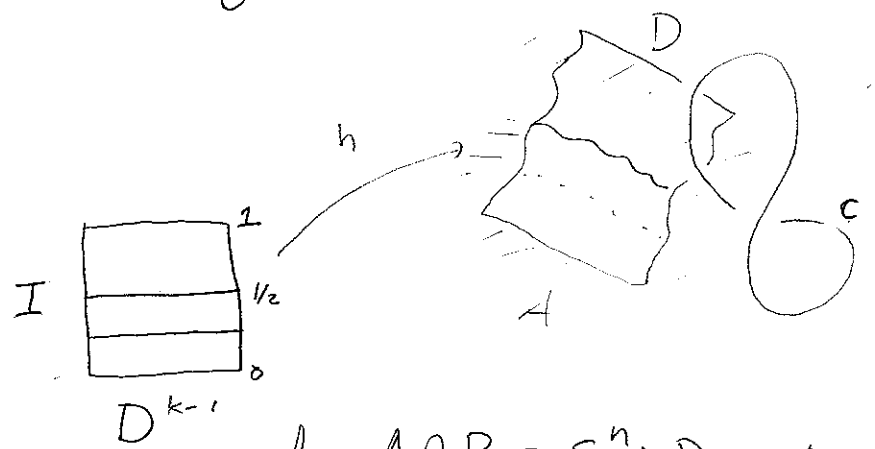
Now consider $S^{n-1} \hookrightarrow S^n$ [Just makes things more sym.]

<u>Cor</u>: $f: S^{n-1} \hookrightarrow S^n$, Then $S^n \setminus f(S^{n-1})$ has two connected components. [Brouwer in 1910, gen J.C.T.]

<u>Thm</u>: a) $h: D^k \hookrightarrow S^n$. Then $\tilde{H}_i(S^n - h(D^k)) = 0$ for all i.

b) $h: S^k \hookrightarrow S^n$ with $k < n$ then $\tilde{H}_i(S^n \setminus h(S^k)) = \begin{cases} \mathbb{Z} & i = n-k-1 \\ 0 & \text{otherwise} \end{cases}$

Proof: a) Induction on k. Case $k=0$ is clear as $S^n \setminus pt = \mathbb{R}^n$. Let $D = h(D^k)$. Suppose not, i.e. $\exists\, \alpha \neq 0$ in $\tilde{H}_i(S^n \setminus D)$, and fix $c \in C_i(S^n \setminus D)$ representing α. Regard $D^k = I^k = D^{k-1} \times I$. [Idea: shrink in final I direction until collapse onto lower dim'l thing]



Let $A = S^n \setminus h(D^{k-1} \times [0, 1/2])$

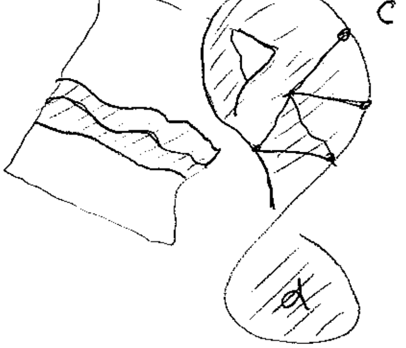
$B = S^n \setminus h(D^{k-1} \times [1/2, 1])$

So $A \cap B = S^n \setminus D$ and $A \cup B = S^n \setminus h(D^{k-1} \times \{1/2\})$

By M-V:

$$\underset{0}{\underset{\parallel}{\tilde{H}_{i+1}(A \cup B)}} \longrightarrow \underset{\alpha}{\tilde{H}_i(A \cap B)} \xrightarrow[\cong]{i_* \oplus -i_*} \tilde{H}_i(A) \oplus \tilde{H}_i(B) \longrightarrow \underset{0\ [\text{by induction}]}{\underset{\parallel}{\tilde{H}_i(A \cup B)}} \longrightarrow$$

So $i_*(\alpha)$ is non-zero in one of $\tilde{H}_i(A)$ and $\tilde{H}_i(B)$. Repeating, can construct nested intervals I_j of len $1/2^j$ s.t. α is $\neq 0$ in $\tilde{H}_i(S^n \setminus h(D^{k-1} \times I_j))$. Let $p = \bigcap I_j$. By induction α is 0 in $\tilde{H}_i(S^n \setminus h(D^{k-1} \times p))$. Let $d \in C_i(S^n \setminus h(D^{k-1} \times p))$

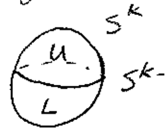


be such that $\partial d = c$.

By compactness, $\exists\, j$ s.t. d is also disjoint from $h(D^{k-1} \times I_j)$

But then $\alpha = 0$ in $\tilde{H}_i(S^n \setminus h(D^{k-1} \times I_j))$ a contradiction. So $\tilde{H}_i(S^n \setminus h(D^k)) = 0\ \forall i$.

b) Again, induct on k. For $k=0$, $S^n \setminus \{\text{two pts}\} \cong S^{n-1} \times \mathbb{R}$ so done. In general, $S^k = U \cup L$ where $U, L \cong D^k$ meeting along S^{k-1}!



Let $A = S^n \setminus h(U)$, $B = S^n \setminus h(L)$ so $A \cap B = S^n \setminus h(S^k)$, $A \cup B = S^n \setminus h(S^{k-1})$.

M-V gives isom $\tilde{H}_i(A \cap B) \cong \tilde{H}_{i+1}(A \cup B)$

as $\tilde{H}_*(A) = 0$ by (a). ■

Lecture 28

Last time:

Thm: a) $h: D^k \hookrightarrow S^n$, then $\tilde{H}_i(S^n \setminus h(D^k)) = 0$ for all i.

b) $h: S^k \hookrightarrow S^n$ with $k<n$. Then $\tilde{H}_i(S^n \setminus h(S^k)) = \begin{cases} \mathbb{Z} & \text{when } i=n-k-1 \\ 0 & \text{otherwise.} \end{cases}$

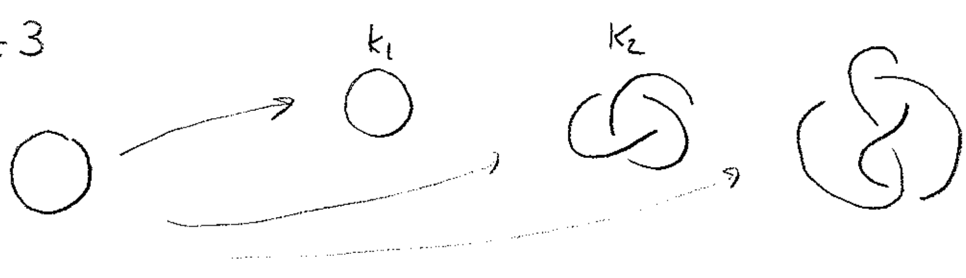
Cor: If $X \subseteq S^n$ is $\cong$ to S^{n-1} then $S^n \setminus X$ has two connected components. [Draw Alex Horned Sphere]

[Reiterate how these are remarkable given the wild embeddings]

Thm (Mazur-Brown 1960s) $f: S^{n-1} \hookrightarrow S^n$ which is "locally flat," i.e. extends to an embedding $S^{n-1} \times I \hookrightarrow S^n$, then $\exists\, h: S^n \to S^n$ s.t. $h(f(S^{n-1})) = S^{n-1}$. [IOW, Schöflies Thm holds.]

Not true for $f: S^{n-2} \hookrightarrow S^n$ (knot theory)

Ex: $n=3$



are all diff [homology is useless.]

Can be e.g. by using fund gps, and show $\pi_1(S^3 \setminus k_1) = \mathbb{Z}$, $\pi_1(S^3 \setminus k_2) = \langle x, y \mid x^2 = y^3 \rangle$ is non-ab. by writing down some perm. rep.

Note: X any path conn. space. Then $H_1(X;\mathbb{Z}) \cong (\pi_1(X))^{ab}$
$= \pi_1 X / [\pi_1 X, \pi_1 X]$

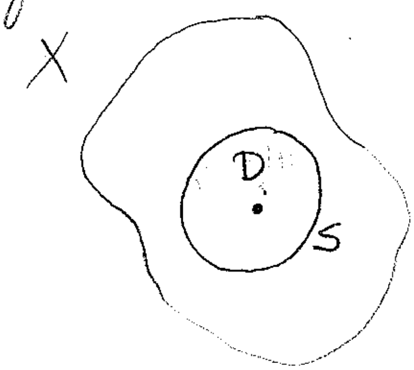
Point: $\pi_1(X) \longrightarrow H_1(X;\mathbb{Z})$ where α is a gen of $H_1(X;\mathbb{Z})$
$\gamma \longmapsto \gamma_*(\alpha)$

[Clearly gives some map $(\pi_1(X))^{ab} \to H_1(X;\mathbb{Z})$, can show is an isom] §2.A.

[Codim 2 is the <u>unique</u> dim where there a <u>diff.</u> locally flat maps $f: S^k \to S^n$. In C^∞ this is not the case — $\exists$ distinct smooth emb. of S^3 in S^6]

Thm: Suppose $X \subseteq \mathbb{R}^n$ is homeo to an open set in $\mathbb{R}^n$. Then X is open. [Invariance of domain]

Pf: Can replace $\mathbb{R}^n$ with S^n. Let $x \in X$. Then x has a nbhd $D \subseteq X$ homeo to D^n, and let S corres. to ∂D^n under this homeo. Now $S^n \setminus D$ is open and connected by Thm. [For open sets in $\mathbb{R}^n$, path conn = loc path conn.] Now $S^n \setminus S$ is open and has two components by Thm, part b.] Now



$$S^n \setminus S = \underbrace{S^n \setminus D}_{\text{conn}} \amalg \underbrace{D \setminus S}_{\text{conn as an open ball.}}$$

$\Rightarrow$ these are the conn. components, hence open in S^n.
So $D \setminus S$ is an open nbhd of $x \subseteq X$. So X is open. ∎

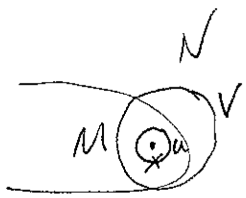
Cor: M cpt n-mfld, N a connected n-mfld. Then an embed. $M \hookrightarrow N$ is surjective.

[So e.g. S^n does not embed in $\mathbb{R}^n$.]

Cor²: $\mathbb{R}^n$ does not contain a subspace $\cong \mathbb{R}^k$ if $k > n$.

Pf: if it did, $\mathbb{R}^n \supseteq S^n$, contradicting Cor.

Pf of Cor: M is closed in N as it is compact and N is Hausdorff. Each pt $x \in M$ has nbhd $U \subseteq M$ and $V \subseteq N$ which are homeo to $\mathbb{R}^n$. We may assume $U \subseteq V \xRightarrow{\text{Thm}} U$ is open in $V \Rightarrow$ U open in N. Then M is open and closed and so is all of N. ∎



Lefschetz Fixed Point Thm: X is a finite Δ-complex. If $f: X \to X$ with $\tau(f) \neq 0$, then f has a fixed pt.

$\varphi: \mathbb{Z}^n \xrightarrow[(a_{ij})]{} \mathbb{Z}^n$ have $\operatorname{tr}\varphi = \sum a_{ii}$ [doesn't depend on choice of basis.]

More gen $\varphi: A \to A \leftarrow$ abelian gp, set $\operatorname{tr}\varphi = \operatorname{tr}\bar{\varphi}$, $\bar{\varphi}: A/\text{torsion} \to A/\text{torsion}$

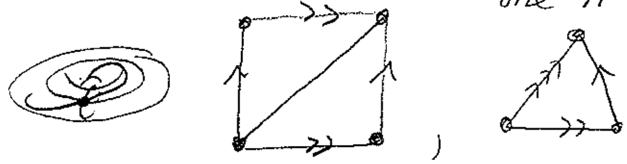
Lefschetz Number: $\tau(f) = \sum (-1)^n \operatorname{tr}\left(f_*: H_n(X) \to H_n(X)\right)$

If $f \simeq id$ then $\tau(f) = \chi(X)$.

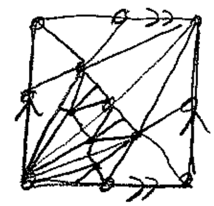
Cor: If X is a contractible finite Δ-complex, $f: X \to X$ then X has a fixed point.

Cor: $f: S^n \to S^n$ has a f.p. unless $\deg(f) = \deg(\text{Antipodal map})$.

<u>Simplical Complex</u>: a Δ-complex where 1) every simplex has distinct vertices 2) Any $n+1$ vertices are the vertices of <u>at most</u> one n-simplex.

<u>No</u>:  <u>Yes</u>: 

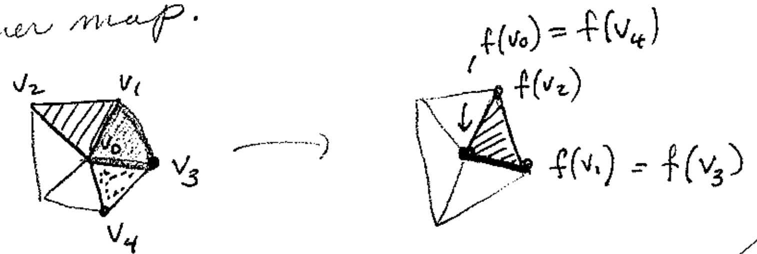
<u>Note</u>: Any Δ complex can be made simplicial if you subdivide twice



[Technically convenient.]

<u>Def</u>: A map $f: K \to L$ of simplicial complexes is <u>simplicial</u> if it sends each simplex of K to a simplex of L by a liner map.

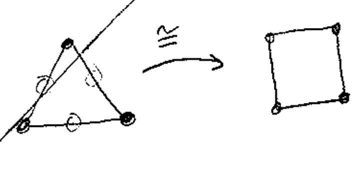
<u>Def</u>. by what it does to vertices.

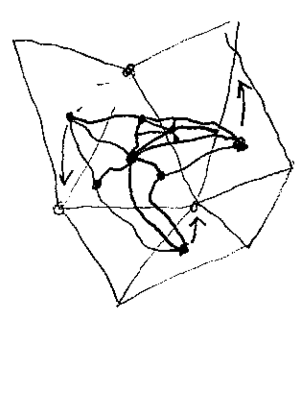


<u>Note</u>: A simplicial map gives an $f_\#: C_n(K) \to C_n(L)$ allowing us to compute f_* on homology.

Didn't get to

<u>Simplicial Approx</u>: If K a finite simplicial complex and L any simplicial complex, then a map $f: K \to L$ is homotopic to a map which is simplicial w.r.t. a repeated subdivision of K.





Lecture 30 Today: Lefshetz Fixed Point Theorem:

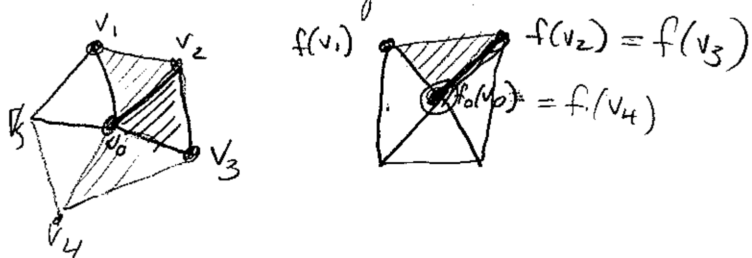
X a finite simplicial complex. $f : X \to X$

Set $\tau(f) = \sum (-1)^n \, tr(f_* : H_n(X;\mathbb{Z}) \circlearrowleft)$

If $\tau(f) \neq 0$ then f has a fixed point

[By doing some point set top, X can be replaced with a CW complex, cpt mfld. Need to be able to compute f_* combinatorially.]

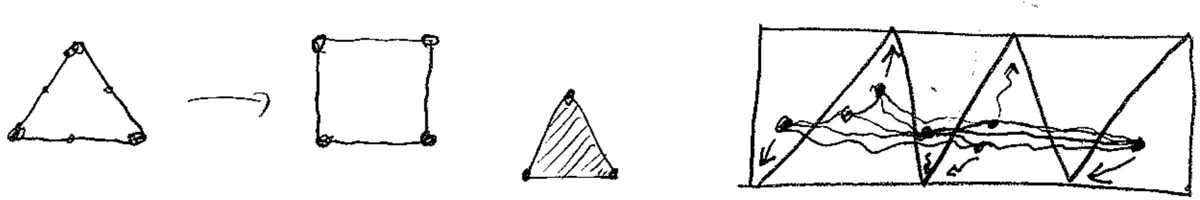
Def: A map $f : K \to L$ of simplicial complexes is simplicial if it sends each simplex of K to a simplex of L by a linear map.



[Det by what it does to $K^0 \to L^0$]

Note: A simplicial map gives $f_\# : C_n^\Delta(K) \to C_n^\Delta(L)$ [chain map] which induces f_* on homology. [Subtle point: what about collapsed simplices.]

Simplicial Approx: $f : K \to L$ of simplicial complexes, with K finite. Then f is homotopic to a map which is simplicial w.r.t. a repeated subdivision of K.



Pf of L.F.P.T.: Suppose $f: X \to X$ has no fixed point

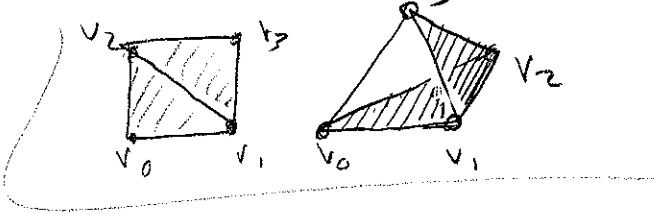
Idea: Suppose f was simplicial. Then no simplex is sent to itself [Q: B.F.P. Thm] Thus the matrix for $f_\#: C_n(X) \circlearrowleft$ has zeros on the diagonal, $\Rightarrow tr(f_\#) = 0 \xRightarrow{Alg} tr(f_*) = 0$.

[Q: what's the problem here? there may not exist such subdivision. $S^1 \longrightarrow S^1$, $z \longrightarrow z^2$ ]

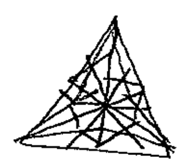
Claim: $\exists$ a subdivision L of X and a subdivision K of L s.t. f is hom to $g: K \longrightarrow L$ simplicial s.t. $\forall \sigma$ of K, $\sigma \cap g(\sigma) = \phi$.

Pf: Consider X as a metric space where each simplex has a euclidean metric. $X \hookrightarrow \Delta^n$

By compactness $\exists \varepsilon > 0$ s.t. $d(x, f(x)) > \varepsilon$.



Let L be a subdivision of X where all simplices have diam $\leq \varepsilon/2$.



Let K be a subdiv of X ~~w/ker~~ ~~w/~~ $g: K \longrightarrow L$ a simp map $\simeq f$.

Point: Construction outlined moves pts at most the the diam of a simplex of L s.o.

$d(f(x),g(x)) \leq \varepsilon/2 \Rightarrow d(g(x),x)) > \varepsilon/2$

$\Rightarrow g(\sigma) \cap \sigma = \emptyset \quad \forall \sigma.$

Why is this good enough? Consider CW homology

$$C_n^{CW}(K) = H_n(K^n, K^{n-1})$$

have

$$H_n(K^n, K^{n-1}) \xrightarrow{f_*} H_n(K^n, K^{n-1})$$

Claim: $\operatorname{tr} f_* = 0$.

[have to argue that this induces the right map on homology]

$H_n(K^n, K^{n-1})$ is gen by $\Delta^n \to 1$

The "coor" of some $\alpha \in H_n(K^n, K^{n-1})$ are read off by collapsing $\to H_n(K^n, K^{n-1} \cup \text{all } \Delta^n\text{'s but one})$

Thus diagonals are zero.

Alg fact:

$$\begin{array}{ccccccccc} 0 & \to & A & \to & B & \to & C & \to & 0 \\ & & \alpha\downarrow & & \beta\downarrow & & \gamma\downarrow & & \\ 0 & \to & A & \to & B & \to & C & \to & 0 \end{array}$$

$\operatorname{tr}\beta = \operatorname{tr}(\alpha) + \operatorname{tr}(\gamma).$