Lecture 12 HW: also look up homotopy stuff.
§1.3. #23, 25 §2.1. #1, 5, 8, 10(a). (final version.)

π_1: fine so far as it goes, but need higher dimensional invariants.

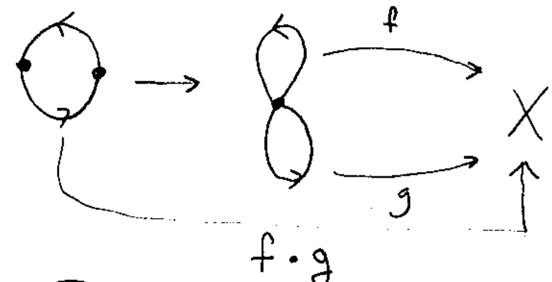
[can't distinguish S^2 from S^3, det by $X^{(2)}$ etc.]

__Higher homotopy groups:__

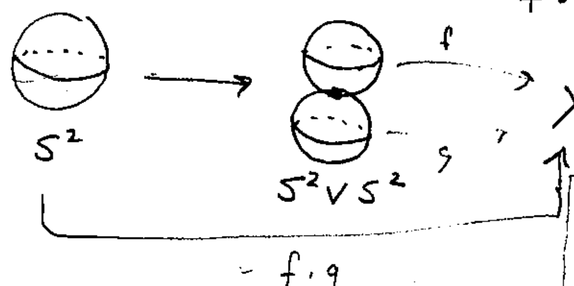
$\pi_n(X, x_0)$ = homotopy class of maps $(S^n, *) \longrightarrow (X, x_0)$

[don't mention, but, homotopies must pres. base pts.]

__Opperation:__ for π_1:



for π_2:



[same for higher n]

[Like π_1 these are homotopy invariant]

__Problem:__ Hard to compute:

$\pi_1 S^2 = 1$
$\pi_2 S^2 = \mathbb{Z}$ gen id
$\pi_3 S^2 = \mathbb{Z}$ $S^3 \longrightarrow S^2$ Hopf fibration
$\pi_4 S^2 = \mathbb{Z}/2$
$\pi_5 S^2 = \mathbb{Z}/2$
$\vdots$ $\mathbb{Z}/12$

$\exists$ an algorithm to compute $\pi_n S^m$.

Even for π_1 answer may not be so useful, even for CW complexes.

__Issue:__ Any reasonable question about finitely presented groups is undecideable.

Ex:  $\not\cong$ 

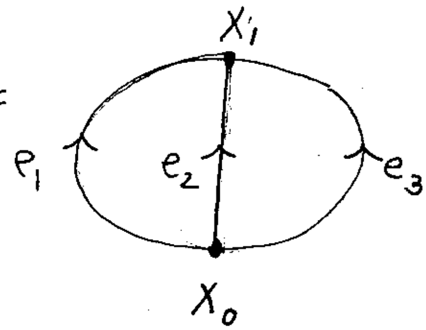
π_1 : $\langle a,b \mid aba^{-1}b^{-1}=1\rangle$, $\langle a,b,c,d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1}=1\rangle$

$$\pi_1^{ab} = \pi_1 / [\pi_1,\pi_1] = \mathbb{Z}^2 \text{ vs. } \mathbb{Z}^4$$

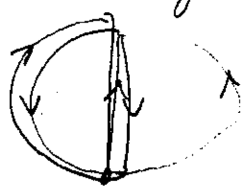
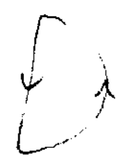
[Higher homotopy will be covered in Winter Qtr's 151b].

Homology: $H_n(X) = \dfrac{n \text{ dim'l things w/o boundaries}}{\text{boundaries of } n+1 \text{ dim'l things}}$

Ex: $H_1(X) = \pi_1^{ab}(X)$

$X =$ 

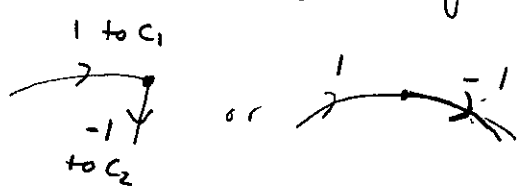
after abelianization

 $=$ 

$(0,-1,1)$

$\gamma \in \pi_1(X)$ get #s c_1, c_2, c_3 corr to how many times we cross each edge w/ sign.

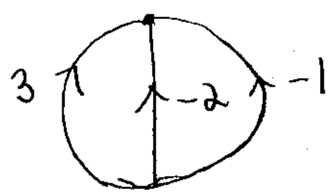
not all triples appear



thus

$$c_1+c_2+c_3=0$$

This is sufficient, too.

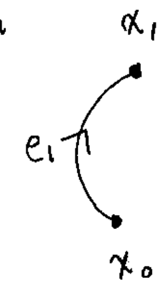


X space w/ a cell decomp.

<u>n-chains:</u> $C_n(X)$ free abelian gp gen by the n-cells.

$$C_0(X) = \mathbb{Z} \oplus \mathbb{Z} = \{a_0 x_0 + a_1 x_1\}$$
$$C_1(X) = \mathbb{Z}^3 = \{c_1 e_1 + c_2 e_2 + c_3 e_3\}$$
$$C_n(X) = 0 \quad n > 1.$$

<u>Boundary map:</u> $\partial_n : C_n(X) \to C_{n-1}(X)$ a homomorphism



$\partial_1 : C_1(X) \to C_0(X)$ $\quad \partial_1(e_1) = x_1 - x_0$

$\partial_1(e_i) = x_1 - x_0$

<u>Cycle:</u> $c \in C_n(X)$ w/ $\partial c = 0$, $\ker \partial_n$

<u>0-cycles:</u> $\ker \partial_0 = C_0(X)$.

<u>1-cycles:</u> $\partial_1(c_1 e_1 + c_2 e_2 + c_3 e_3) = c_1(x_1 - x_0) + c_2(x_1 - x_0) + c_3(x_1 - x_0)$

$$= (c_1 + c_2 + c_3) x_1 - (c_1 + c_2 + c_3) x_0$$

so $\ker \partial_1$ = those w/ $c_1 + c_2 + c_3 = 0$.

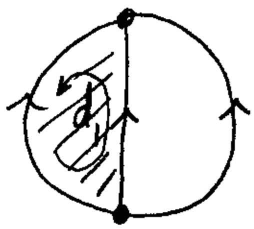
[these are the n-dim'l things w/o boundries]

$$\underline{H_n(X)} = \ker \partial_n / \operatorname{im} \partial_{n+1} .$$

$$H_0(X) = C_0(X) / c x_1 - c x_2 = \mathbb{Z}^2 / \langle (1,-1) \rangle \overset{\text{change of basis}}{=} \mathbb{Z}.$$

$$H_1(X) = {}^{\ker \partial_1}\!/_{\mathrm{im}\, \partial_2} = \ker \partial_1 = \{\text{those w/ } c_1 + c_2 + c_3 = 0\} = \mathbb{Z}^2 \text{ w/ basis } e_1 - e_2,\ e_2 - e_3.$$

More complicated:



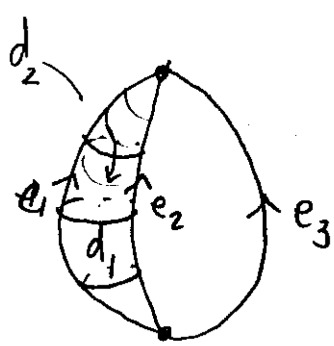
$C_2(X) = \mathbb{Z}$ gen by d_1

$\partial_2(d_1) = e_2 - e_1$

$$H_1(X) = {}^{\ker \partial_1}\!/_{\mathrm{im}\, \partial_2} = \mathbb{Z} \quad \text{gen by image of } e_2 - e_3$$

$H_2 = 0$ as $\ker \partial_2 = 0$.

Finally



$H_0 = \mathbb{Z}$

$H_1 = \mathbb{Z}$

$H_2 = \mathbb{Z}$ gen by $d_1 - d_2$

$H_n = 0 \quad n > 2.$

Lecture 13

Last time: Homology 101

X a CW complex, $C_n(X)$ free abelian gp w/ basis the n-cells of X

$$\to C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial_{n-1}}$$

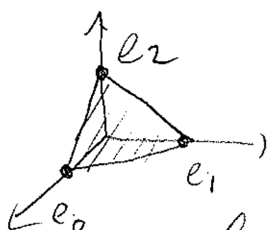
$$H_n(X) = \ker \partial_n / \operatorname{im} \partial_{n+1}$$

[Hard part: defining boundary maps.

Solution: Use a restricted class of CW complexes, called Δ-complexes.]

n-simplex: $n=0$ 1 2 3 ... [convex hull of $n+1$ generic pts in $\mathbb{R}^m$, $m \geq n+1$]

Standard n-simplex: Δ^n = convex hull of $e_0, \dots, e_n$ in $\mathbb{R}^{n+1}$ $= \left\{ (t_0, \dots, t_{n+1}) \in \mathbb{R}^n \;\middle|\; \sum t_i = 1,\ t_i \geq 0 \right\}$

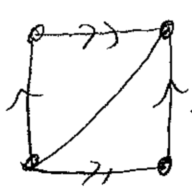
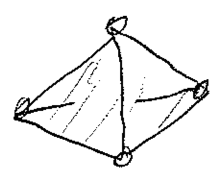


A face of a simplex is a subsimplex spanned by all but one of vertices.

The boundary $\partial\Delta$ is the union of all the faces. The interior

$$\mathring{\Delta} = \Delta \setminus \partial\Delta$$

[Now define a new class of CW complexes, where the cells are simplices and attaching maps glue the faces of Δ^n to images of Δ^{n-1} via nice maps.]

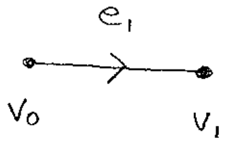
Ex: $T =$  $=$  or $S^2 =$  $= \partial\Delta^3$

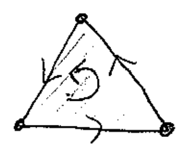
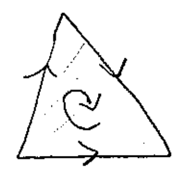
Def: [X a top space] A Δ-complex structure on X is a collection of maps $\sigma_\alpha : \Delta^n \to X$ [n deps on α] s.t.

1) $\sigma_\alpha|_{\mathring{\Delta}^n}$ is injective. Every $x \in X$ is in exactly one $\sigma_\alpha(\mathring{\Delta}^n)$

2) If F is a face of Δ^n, then $\sigma_\alpha|_F$ is one of $\sigma_\beta : \Delta^{n-1} \to X$ after we ident F with Δ^{n-1} via a linear homeo.

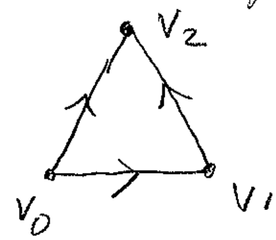
3) $A \subseteq X$ is open $\Longrightarrow$ $\sigma_\alpha^{-1}(A)$ is open in Δ^n for each α.

Orientations: 

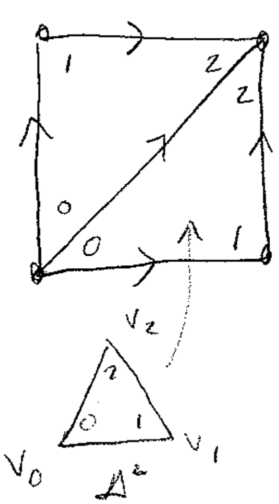
$\partial e_1 = v_1 - v_0$  vs. 

An oriented simplex is a simplex Δ w/ an ordering of its vertices $[v_0, v_1, \ldots, v_n]$ [Pins down an ident of Δ w/ standard Δ^n.]

A face of an orient simplex has the orient from restricting the order.



Addendum 2') The ident of F with Δ^{n-1} pres the order of the vertices.

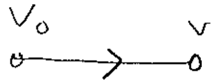
$T^2 =$ 

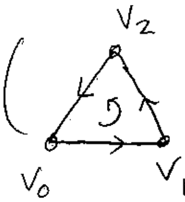
[This is a rest. over what we had before, but not an important one this term (important when def. cup product on homotops)]

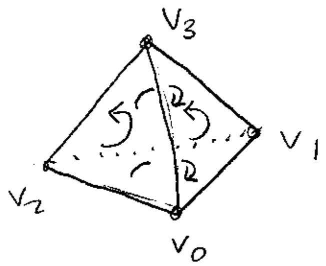
(non ex )

Homology of X w/ Δ-complex structure

$C_n(X)$ - free ab gp gen by $\sigma_\alpha : \Delta^n \to X$.

$\partial\Big($  $\Big) = [v_1] - [v_0]$

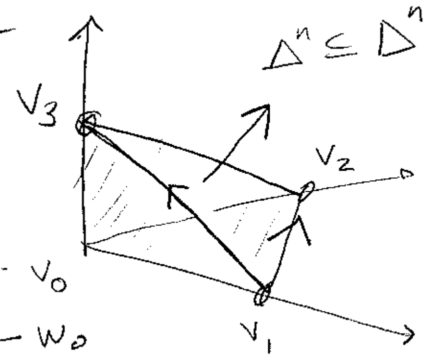
$\partial\Big($  $\Big) = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$

$\partial\Big($  $\Big) = [v_1, v_2, v_3] - [v_0, v_2, v_3] + [v_0, v_1, v_3] - [v_0, v_1, v_2]$

[What is an orientation of $\mathbb{R}^n$? A choice of basis, with two equal iff det(change of basis) > 0.]

Where do the signs come from

$F = [w_0, \ldots, w_{n-1}]$



sign is + iff

(outward normal to F, $w_1 - w_0, \ldots, w_{n-1} - w_0$

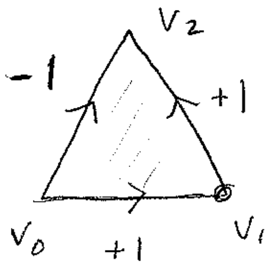
is a pos basis for $\mathbb{R}^n$

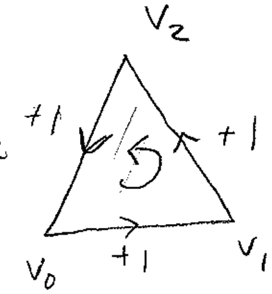
Def: $\partial_n(\sigma_\alpha) = \sum_{i=0}^{n} (-1)^i \sigma_\alpha \Big|_{[v_0, \ldots, \hat{v}_i, \ldots, v_n]}$

↑ means omit

$\partial_0 = 0$ map

Lemma: $\partial_{n-1} \circ \partial_n = 0$

Ex: 

think 

Moral: Really just a statement about the standard $(n+1)$ simplex, namely $\partial \Delta^{n+1} \cong S^n$ has no boundary

Pf: [If time allows] Suffices to check on a basis that:

$$\partial_{n-1}(\partial_n(\sigma_\alpha)) = \partial_{n-1}\left(\sum_{n=0}^{n} (-1)^i \sigma_\alpha|_{[v_0, \dots, \hat{v}_i, \dots, v_n]} \right)$$

$$= \sum_{n=0}^{n} (-1)^i \left(\sum_{j=0}^{i-1} (-1)^j \sigma_\alpha|_{[v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots v_n]} + \sum_{j=i+1}^{n} (-1)^{j-1} \sigma_\alpha|_{[v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_n]} \right)$$

$= 0$ [as each $\sigma_\alpha|_{[w_0, \dots, w_{n-2}]}$ appears twice w/ opp. signs. ■

Cor: $\operatorname{im}(\partial_n) \subseteq \ker(\partial_{n-1})$ **Pf:** $C_{n+1}(X) \to C_n(X) \xrightarrow{\partial_{n-1}} C_{n-1}(X)$

Upshot: Can define $H_n(X) = \ker \partial_n / \operatorname{im} \partial_{n+1}$

Q: Does this depend on the choice of Δ complex structure?

Q: How do we define H_n for a general X?

Lecture 14

Last time: Simplicial Homology

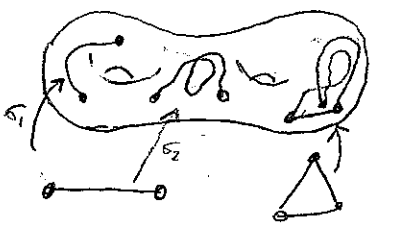
Δ-complex str for X: $\sigma_\alpha : \Delta^n \to X$ cells

Homology of X w.r.t. Δ-str: $H_n^\Delta(X)$ [new decoration]

Today: How to define homology for all X [Will use to show is indep of the Δ-str.]

X a top space [not nec. a CW-complex]

A <u>singular n-simplex</u> is a map $\sigma : \Delta^n \to X$



$C_n(X) =$ free abelian gp gen by all sing n-simplices [very large!]

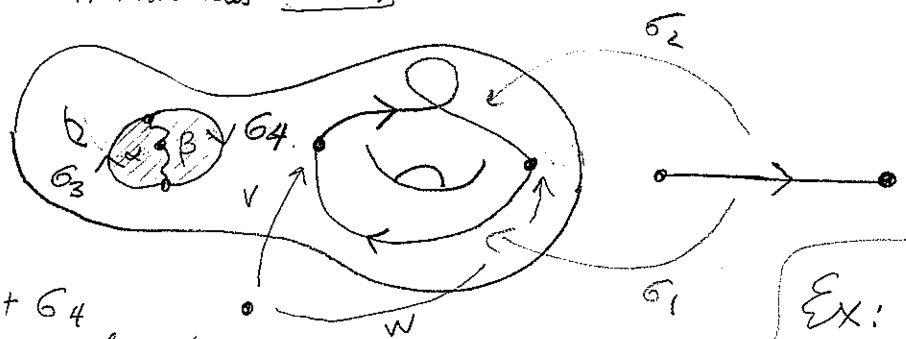
$\partial_n : C_n(X) \to C_{n-1}(X)$ by $\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma\Big|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}$

thought of as a map $\Delta^{n-1} \to X$ under the ordered vertex pres linear homeo of [ ] and Δ^{n-1}

<u>Lemma</u>: $\partial_n \circ \partial_{n+1} = 0$, so can define

$$H_n(X) = \overbrace{\ker \partial_n}^{n\text{-cycles}} \Big/ \underbrace{\operatorname{im} \partial_{n+1}}_{n\text{-boundaries}} \quad \text{— } \underline{\text{singular homology of } X}$$

<u>Ex</u>:



$\partial \sigma_1 = v - w$, [not a 1-cycle]

$\sigma_1 + \sigma_2$ is a 1-cycle

$\sigma_3 + \sigma_4$ is a 1-boundary $= \partial(\alpha + \beta)$

<u>Ex</u>: $X = \{pt\}$

[Query: How many singular n-simplices?]

$\partial \sigma_1 = \sigma_0 - \sigma_0 = 0$

$\exists! \; \sigma_n : \Delta^n \to X$.

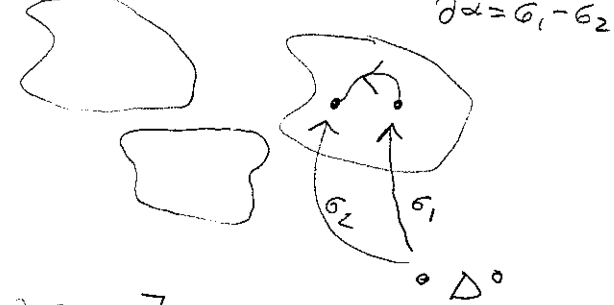
$$C_4(X) \longrightarrow C_3(X) \xrightarrow{0} C_2(X) \xrightarrow{\cong} C_1(X) \xrightarrow{0} C_0(X) \xrightarrow{\partial_0} 0$$

$C_1(X) = \mathbb{Z}$, $C_0(X) = \mathbb{Z}$

Thus $H_0(X) = \mathbb{Z}$, $H_n(X) = 0$ for $n \geq 1$. [Point out calc in Δ-complex case]

For any Y, will show $H_n^{\Delta}(Y) \cong H_n(Y)$ [thus the former is indep of Δ-complex str.]
[will take 2 weeks to build up enough prop of H_n to show this.]

<u>Prop</u>: $H_0(X) = \bigoplus_{\substack{\text{path comp} \\ \text{of } X}} \mathbb{Z}$ <u>Idea</u>:



$f: X \to Y$ a map. [Have an induced map on π_1]

Will define $f_*: H_n(X) \to H_n(Y)$ as follows.

draw first

$$\begin{tikzcd}
C_{n+1}(X) \arrow[r, "\partial_{n+1}"] \arrow[d, "f_\#"] \arrow[dr, phantom, "\circlearrowright"] & C_n(X) \arrow[r, "\partial_n"] \arrow[d, "f_\#"] \arrow[dr, phantom, "\circlearrowright"] & C_{n-1}(X) \arrow[r] \arrow[d, "f_\#"] & {} \\
C_{n+1}(Y) \arrow[r, "\partial'_{n+1}"] & C_n(Y) \arrow[r, "\partial'_n"] & C_{n-1}(Y) &
\end{tikzcd}$$

$H_n(X) = \ker \partial_n / \operatorname{im} \partial_{n+1}$

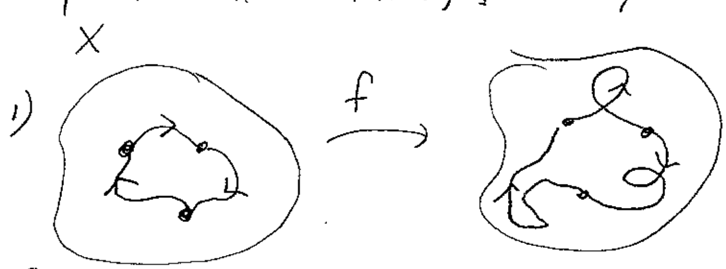
$H_n(Y) = \ker \partial'_n / \operatorname{im} \partial'_{n+1}$

where $f_\#(\sigma: \Delta^n \to X) = (f \circ \sigma): \Delta^n \to Y$. To get a map on $H_n(X)$ need

1) $f_\#(\ker \partial_n) \subseteq \ker \partial'_n$ [gives map $\ker \partial_n \to H_n(Y)$]

2) $f_\#(\operatorname{im} \partial_{n+1}) \subseteq \operatorname{im} \partial'_{n+1}$.

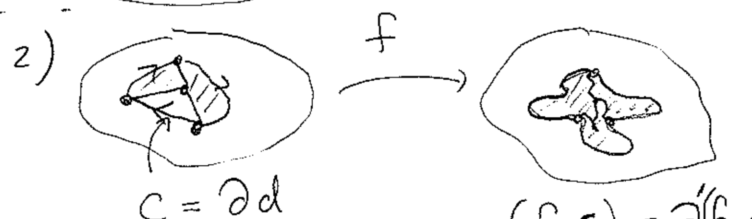
Geometrically, this makes sense



<u>Claim</u>: $f_\# \circ \partial_n = \partial'_n \circ f_\#$

(say: "the diagram commutes")

[Alg, this implies both 1) and 2)]



$(f_\# c) = \partial'(f_\# d)$

$f_\#(\partial d) = \partial'(f_\# d)$

<u>Check</u>: $f_\#(\partial_n \sigma) = f_\#\left(\sum (-1)^i \sigma \big|_{i^{th}\text{ face of }\Delta^n}\right) = \sum (-1)^i \underbrace{f_\#(\sigma)}_{f\circ\sigma}\big|_{i^{th}\text{ face}}$

$= \sum (-1)^i f\circ\sigma \big|_{i^{th}\text{ face}} = \partial_n'\big(f_\#(\sigma)\big)$

<u>Terminology</u>: chain complex: (C_n abelian gps)

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f"] \arrow[dr, phantom, "\circlearrowright"] & C_n \arrow[r, "\partial_n"] \arrow[d, "f"] \arrow[dr, phantom, "\circlearrowright"] & C_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d, "f"] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "\partial_{n+1}'"'] & D_n \arrow[r, "\partial_n'"'] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

w/ $\partial^2 = 0 \Rightarrow$ gives homology groups H_n

f makes diagram commute, "a chain map"

$\Rightarrow$ get a map on homology.

<u>Next time</u>: $f: X \to Y$ a hom. equiv. Then

$f_*: H_n(X) \to H_n(Y)$ is an isom [for each n]

<u>Cor</u>: $H_n(\mathbb{R}^k) = 0 \quad n \geq 1, \quad H_0(\mathbb{R}^k) = \mathbb{Z}$

Also did pf of

from

<u>Thm</u>: $f, g: X \to Y$ are homotopic. Then

$f_* = g_*$

Lecture 15

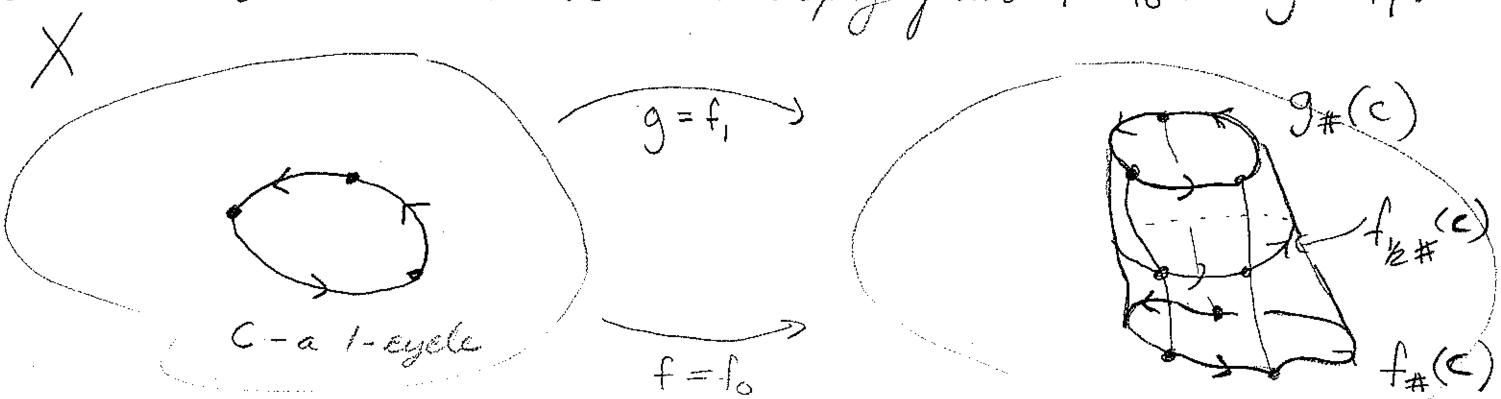
Last time: Singular homology

$C_n(X) =$ free ab gp on all maps $\sigma : \Delta^n \to X$.

Today: *Thm:* $f, g : X \to Y$ homotopic maps, then $f_* = g_* : H_n(X) \to H_n(Y)$.

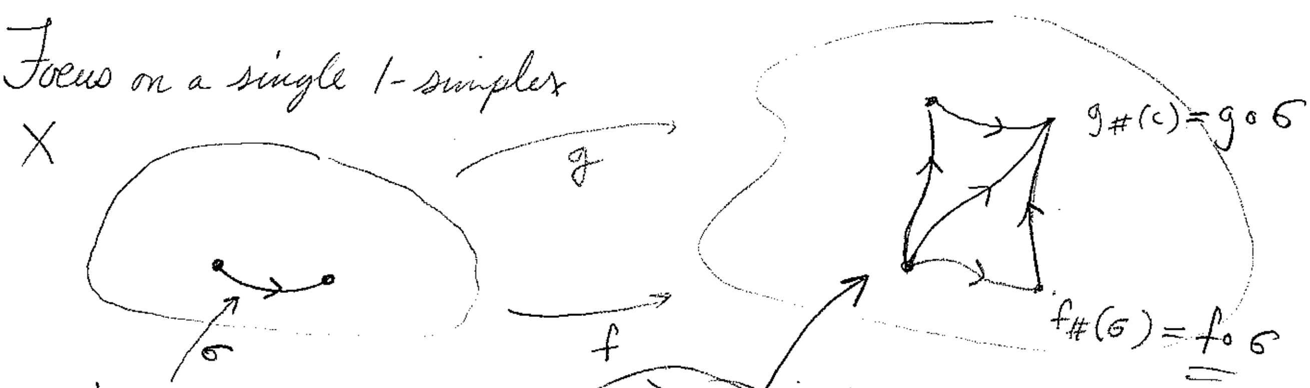
[Saw how this means that hom. equiv spaces have the same homology, so $H_*(\mathbb{R}^k) = H_*(pt)$.]

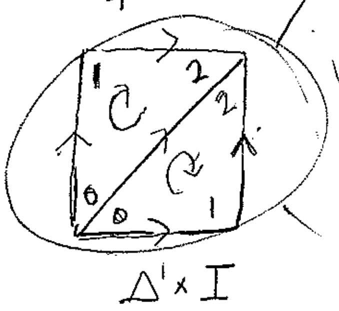
Pf: Let $f_t : X \times I \to Y$ be a homotopy from $f = f_0$ to $g = f_1$.



Need: $f_\#(c) = g_\#(c)$ in $H_1(Y)$, i.e. $\exists\, d \in C_2(Y)$ w/ $\partial d = g_\#(c) - f_\#(c)$

Focus on a single 1-simplex





Have homotopy from $f \circ \sigma$ to $g \circ \sigma$

via $(x,t) \longmapsto f_t(\sigma(x))$

Is this right??

$\Rightarrow$ gives a 2-chain $P(\sigma)$

Note: $\partial P(\sigma) = g_\#(\sigma) - f_\#(\sigma) - P(\partial\sigma)$

Will show such exists for all σ.

Algebra Lemma:

$$\begin{tikzcd}
 & C_{n+1}(X) \arrow[r] \arrow[dl, "P"'] \arrow[d, shift right, "f_\#"'] \arrow[d, shift left, "g_\#"] & C_n(X) \arrow[r, "\partial"] \arrow[dl, "P"'] \arrow[d, shift right, "f_\#"'] \arrow[d, shift left, "g_\#"] & C_{n-1}(X) \arrow[dl, "P"'] \arrow[d, "f_\# \; g_\#"] \\
 {} & C_{n+1}(Y) \arrow[r] & C_n(Y) \arrow[r, "\partial"] & C_{n-1}(Y)
\end{tikzcd}$$

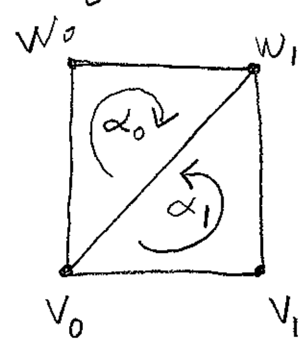
Where $\partial \circ P + P \circ \partial = g_\# - f_\#$. Then $f_\#$ and $g_\#$ induce the same map on homology.

Pf: $c \in C_n(X)$ an n-cycle. Then $g_\#(c) - f_\#(c) = \partial P(c) + \underbrace{P(\partial c)}_{= 0} = \partial P(c)$ (exp. previous picture)

$\Rightarrow g_\#(c) = f_\#(c)$ in $H_n(X)$. ∎

Constructing P: $\sigma : \Delta^n \to X$, consider $\Delta^n \times I \xrightarrow{F} Y$ the induced hom. between $g_\#(\sigma)$ and $f_\#(\sigma)$. $[F = f_t \circ (\sigma \times id_I)]$

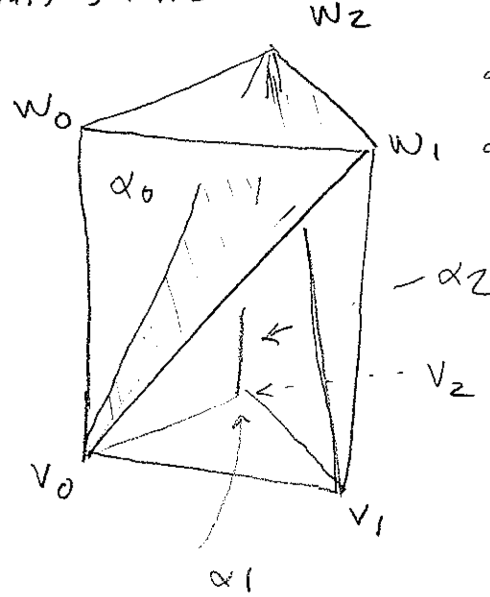
$[v_0, \dots, v_n] = \Delta^n \times \{0\}$ $\quad [w_0, \dots, w_n] = \Delta^n \times \{1\}$



$\Delta^n \times I$ is the union of $(n+1)$-simplices

$$\alpha_i = [v_0, \dots, v_i, w_i, w_{i+1}, \dots, w_n]$$

for $i = 0, \dots, n$.



$\alpha_0 = [v_0, w_0, w_1]$
$\alpha_1 = [v_0, v_1, w_1]$

Now define

$$P(\sigma) = \sum_{i=0}^{n} (-1)^i F|_{\alpha_i}$$

Check: $\partial P(\sigma) = \partial\left(\sum_{i=0}^{n}(-1)^i F|_{\alpha_i}\right) = \sum_{i,\, j\le i} (-1)^i(-1)^j F|_{[v_0,\dots,\hat{v}_j,\dots,v_i,w_i,\dots,w_n]}$

$$+ \sum_{i,\, j\ge i} (-1)^i(-1)^{j+1} F|_{[v_0,\dots,v_i,w_i,\dots,\hat{w}_j,\dots,w_n]}$$

$$P(\partial\sigma) = \sum_{j=0}^{n}(-1)^j P(\sigma|_{[v_0,\dots,\hat{v}_j,\dots,v_n]})$$

$$= \sum_{j,\, i\le j-1}(-1)^{j+i} F|_{[v_0,\dots,v_i,w_i,\dots,\hat{w}_j,\dots,w_n]} + \sum_{j,\, i\ge j+1}(-1)^{j+i+1} F|_{[v_0,\dots,\hat{v}_j,\dots,v_i,w_i,\dots,w_n]}$$

So $P(\partial\sigma) + \partial P(\sigma) = \sum_{i} (-1)^{2i} F|_{[v_0,\dots,v_{i-1},w_i,\dots,w_n]}$

$$+ \sum_{i=i} (-1)^{2i+1} F|_{[v_0,\dots,v_i,w_{i+1},\dots,w_n]}$$

$$= g_\#(\sigma) - f_\#(\sigma)$$

Recall $A \overset{\text{closed}}{\subseteq} X$. If A is contractible, then $X \to X/A$ is a hom. equiv.

Goal: $A \overset{\text{cld}}{\subset} X$, want to relate $H_*(A)$, $H_*(X)$, $H_n(X/A)$

[roughly, will be able to compute one if we know the other two.]

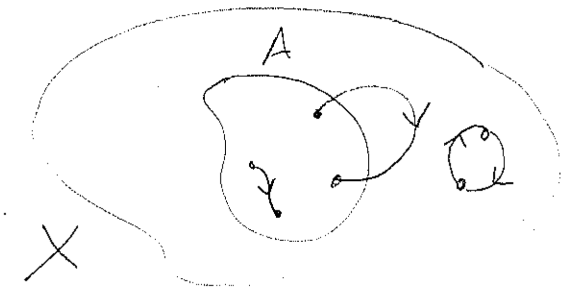
Ex: $X = D^n$, $A = \partial D^n = S^{n-1}$; $X/A \overset{?}{=} S^n$.

Lecture 16

Last time: $f, g : X \to Y$ homotopic.
Then $f_* = g_* : H_n(X) \to H_n(Y)$. [Explain notation]

Goal: For $A \overset{\text{clsd}}{\subseteq} X$, relate $H_*(A)$, $H_*(X)$, $H_*(X/A)$
[Will alow us to compute one if we know the other two.]

Today: Relative homology. [A stand in for]



$$\begin{array}{ccc} C_n(X) & \xrightarrow{\partial} & C_{n-1}(X) \\ \cup\vert & & \cup\vert \\ C_n(A) & \xrightarrow{\partial} & C_{n-1}(A) \end{array}$$

draw on now

[Ex. of relative cycles]

New chain complex: $C_n(X,A) = C_n(X)/C_n(A)$ w/ induced ∂ map

Relative Homology: $H_n(X,A)$ [The homology of the chain complex]

elts rep by *relative cycles*: $c \in C_n(X)$ s.t. $\partial c \in C_{n-1}(A)$

$c = 0$ in hom $\Leftrightarrow$ *relative boundary*: $c = \partial d + a$ where $d \in C_{n+1}(X)$, $a \in C_n(a)$

Ex: $\bullet \longrightarrow \bullet$

Compute using *simplicial* homology

$X = I$, $A = \partial I$, $X/A = S^1$

$H_0(X,A) \overset{?}{=} 0$ [!]

$H_1(X,A) = \mathbb{Z}$ [All others are 0]

Exact Sequences:

$\cdots \to A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \xrightarrow{\alpha_{n-1}} \cdots$ (gps) is *exact* if $\ker \alpha_n = \operatorname{im} \alpha_{n+1}$.

[Query class on famil. with this concept. How is it diff from a chain complex?]

[?]

Ex: $0 \to A \xrightarrow{\alpha} B$ is exact $\Leftrightarrow$ α is 1-1

$B \xrightarrow{\beta} C \to 0$ is exact $\Leftrightarrow$ β is onto.

Thus $0 \to A \to B \to C \to 0$ is exact means B has a normal subgp A w/ quotient C. ["short exact seq"].

<u>Thm</u>: $A \subseteq X$. Then the following is exact:

$$\cdots \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(X) \to$$
$$\cdots \to H_0(A) \to H_0(X) \to H_0(X,A) \to 0.$$

where i_* is induced by inclusion $A \to X$

j_* is induced by $C_n(X) \to C_n(X,A)$

∂ induced by $\left(\substack{\text{rel cycle} \\ \text{in } C_n(X,A)}\right) \xrightarrow{\partial} C_{n-1}(A)$

<u>Homological Algebra</u>: $0 \to C_*(A) \xrightarrow{i_\#} C_*(X) \xrightarrow{j_\#} C_*(X,A) \to 0$

chain maps

A short exact sequence of chain complexes $0 \to A_* \xrightarrow{i} B_* \xrightarrow{j} C_* \to 0$ gives a long exact sequence in homology:

$$\cdots \longrightarrow H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \to \cdots$$

<u>Def of ∂</u>: $[c] \in H_n(C)$

Pick $b \in B_n$ with $j(b) = c$.

Now $\partial b \in \ker j$ [as $\partial c = 0$]

By exactness, $\exists\, a \in A_{n-1}$ w/ $i(a) = \partial b$.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\arrow[r] & A_{n+1} \arrow[r, "\partial"] \arrow[d, "i"] & A_n \arrow[r, "\partial"] \arrow[d, "i"] & A_{n-1} \arrow[r] \arrow[d, "i"] & {} \\
\arrow[r] & B_{n+1} \arrow[r, "\partial"] \arrow[d, "j"] & B_n \arrow[r, "\partial"] \arrow[d, "j"] & B_{n-1} \arrow[r] \arrow[d, "j"] & {} \\
\arrow[r] & C_{n+1} \arrow[r, "\partial"] \arrow[d] & C_n \arrow[r, "\partial"] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & {} \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

a

$b \mapsto \partial b$

c

<u>Set</u>: $\partial([c]) = [a] \in H_{n-1}(A)$

[Q: as $i(\partial a) = \partial(ia) = \partial(\partial b)) = 0$]

Well def: • a only dep on b, as i is 1-1.

• if b' also has $j(b')=c$, $\exists\, d \in A_n$ with $i(d)=b'-b$

Thus
$$\begin{tikzcd} d \arrow[d] \arrow[rr, bend left] & & a-a' \arrow[d, "i"] \\ b-b' \arrow[r] & \partial b-\partial b' & \end{tikzcd}$$
and so $\partial d = a-a'$ as i is 1-1.

So $[a]=[a']\in H_{n-1}(A)$

• if replace c w/ $c+\partial d$, pick b'' with $j(b'')=d$

then $j(b+\partial b'')=c+\partial d$ so can take

$$\begin{tikzcd} b'' \arrow[r, mapsto] \arrow[d, mapsto] & \partial b'' \\ d \arrow[r] & c+\partial d \end{tikzcd}$$

$b'=b+\partial b''$. But $\partial b'=\partial b$ so this doesn't change anything. (Whew!)

Pf of exactness: [Skip some of this as appropriate]

At $H_n(B)$: $\boxed{\operatorname{im} i_* \subseteq \ker j_*}$: $j\circ i$ at chain level

$\boxed{\ker j_* \subseteq \operatorname{im} i_*}$: $b\in B_n$ a cycle in $\ker j$.

So $j(b)=\partial c'$ for some c'. Choose $b''\in B_{n+1}$ w/

$j(b'')=c'$. Thus $\partial b''-b$ is in the kernel of j

Let $d\in A_n$ have $i(d)=\partial b''-b$. Then d is a cycle

and $i_*[d]=[\partial b''-b]=[b]$.

If time remains, can do some other cases.

At $H_n(C)$: $\operatorname{Im} j_* \subseteq \ker\partial$: clear from def. $\ker\partial \subseteq \operatorname{Im} j_*$

At $H_n(A)$: $\operatorname{Im}\partial \subseteq \ker(i_*)$ clear from def.

Lecture 17

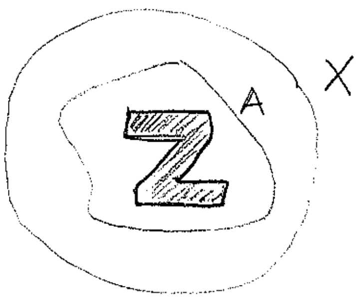
Last time: $A \subseteq X$. The following is exact

$$\to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Goal: (X,A) a "good" pair. Then $H_n(X/A) = H_n(X,A)$ for $n > 1$.
[= clsd + nbhd which def retracts.]

Today: Excision. [Key tool which lets us show]

Excision: $Z \subseteq A \subseteq X$, with $\overline{Z} \subseteq \mathrm{int}(A)$. [Largest open set $\subseteq A$.] Then inclusion induces an isomorphism $H_*(X-Z, A-Z) \to H_*(X,A)$



Equivalently, $A, B \subseteq X$ whose interiors cover X. Then $H_*(B, A\cap B) \to H_*(X,A)$ is an isomorphism.

Reason these are equiv: [Query] $B = X - Z$; $\overline{Z} \subseteq \mathrm{int}(A) \Rightarrow$
$X = (\underbrace{X - \overline{Z}}_{\subseteq \mathrm{int}(B)}) \cup \mathrm{int}(A)$ as desired

Conversely, take $Z = X - B$ then $\overline{Z} = X - \mathrm{int}(B) \subseteq \mathrm{int}(A)$

Jump up and down about excision.

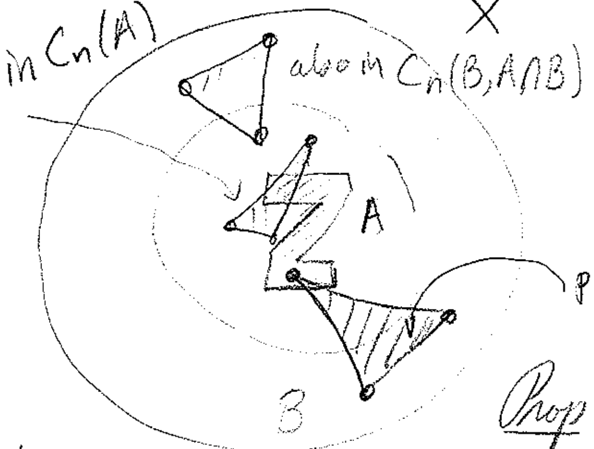
Setup: [More general] $\mathcal{U} = \{U_i\}$ with $\bigcup \mathrm{int}(U_i) = X$.

$C_n^{\mathcal{U}}(X) \leq C_n(X)$ gen by $\sigma: \Delta^n \to X$ w/ image $\subseteq$ in some U_i.

[This is a ~~subcomplex~~, so]: $H_n^{\mathcal{U}}(X)$ the hom of $C_n^{\mathcal{U}}(X)$.

Our case $\mathcal{U} = \{A, B = X - Z\}$

Why: [is $H^{\mathcal{U}}$ relevant?]



$$C_n(B, A\cap B) \xrightarrow{\;i\;} C_n(X,A)$$

[Should be $\cong$ on H_*]

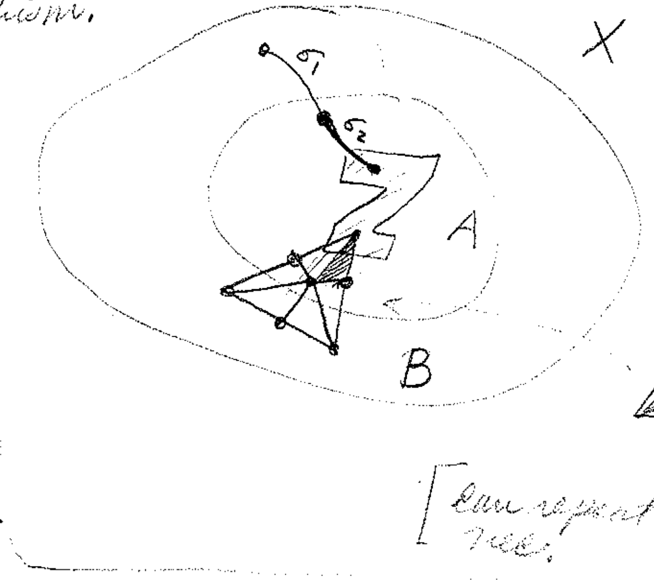
need ← but really only makes sense on $C_n^{\mathcal{U}}(X,A)$

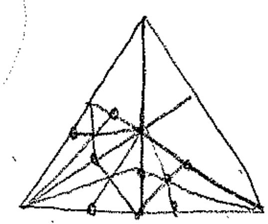
[Leads to question: Do $C_n(X,A)$ and $C_n^{\mathcal{U}}(X,A)$ have the same hom?]

[Not relative]

Prop: $i : C_*^{\mathcal{U}}(X) \hookrightarrow C_*(X)$ is a chain homotopy equivalence. That is $\exists\, \rho : C_*(X) \longrightarrow C_n^{\mathcal{U}}(X)$ s.t. $\rho \circ i$ and $i \circ \rho$ are chain hom. to the identity. In particular $H_n^{\mathcal{U}}(X) \longrightarrow H_n(X)$ is an isomorphism.

Basic idea behind ρ:



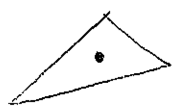


[can repeat if nec.]

[Barycentric subdivision:]

$\delta = [v_0, v_1, \dots, v_n]$ an n-simplex

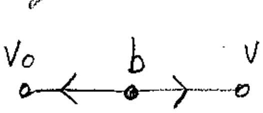
$\delta = \{\sum_{i=0}^{n} t_i v_i \mid \sum t_i = 1\}$

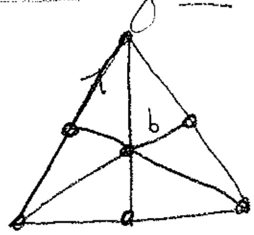


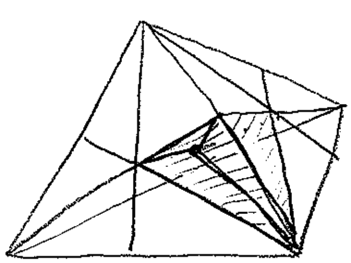
The barycenter of δ: $\underbrace{\frac{1}{n+1}\sum v_i}_{= b}$ [center of mass.]

The barycentric subdivision of δ:
1) Subdivide every face of δ
2) For each $n-1$ simplex $[w_0, \dots, w_{n-1}]$ in $\partial\delta$, $[b, w_0, \dots, w_{n-1}]$ is an n simplex in the subdivision.

Ex:







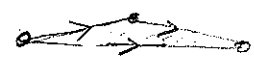
[signs that I won't worry about. I asked this as def. induct by coning.]

<u>Subdivision Opp:</u> $S : C_n(X) \to C_n(X)$

$$\sigma \longmapsto \sum_{\substack{\tau \text{ in barycentre} \\ \text{subdivision of } \Delta^n}} \pm \, \sigma|_\tau .$$

This is a chain map because of the inductive nature of the def.

<u>Fact</u>: S is chain hom. to the ident



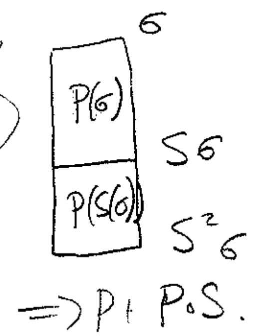
(See Hatcher 119–122)

<u>Idea</u> of proof of Prop: For each σ, $S^n(\sigma) \in C_n^{\mathcal{U}}(X)$ for some n. If n were uniform then

$S^n : C_*(X) \to C_*^{\mathcal{U}}(X)$ would be the needed chain homotopy equivalence.

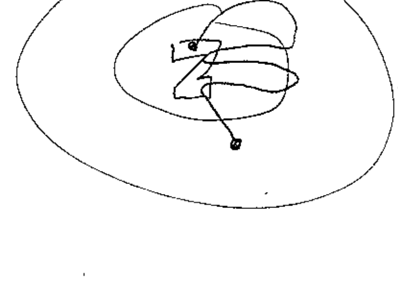
[The above was over way too fast. Also covered.

1) S^n is chain hom to ident by $\sum_{0 \le i \le m} P S^i$ where P is the chain hom for S.



2) $m(\sigma)$ minimal m s.t. $S^{m(\sigma)}(\sigma) \in C_n^{\mathcal{U}}(X)$

$\exists$ as <u>interiors</u> of A and B cover X.



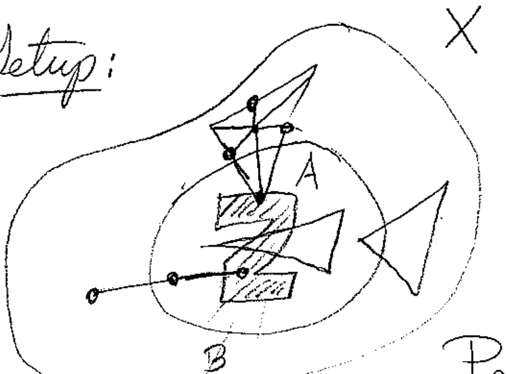
Lecture 18

HW#7: §2.1: 12, 15, 17, 21, 22, 29

Today: Rest of the proof of

Excision: $Z \subseteq A \subseteq X$ w/ $\overline{Z} \subseteq \mathrm{int}(A)$. Then
$$H_n(X-Z, A-Z) \xrightarrow{i_*} H_n(X,A) \text{ is an isom.}$$

Setup:



$\mathcal{U} = \{A, B = X - Z\}$ [int. cover]

$C_n^{\mathcal{U}}(X) \leq C_n(X)$
gen by $\sigma : \Delta^n \to X$ w/ image in A or B

Prop: $C_n^{\mathcal{U}}(X) \xrightarrow{i} C_n(X)$ is a chain homotopy equivalence. [we're in the middle of proving this.]

$S : C_n(X) \longrightarrow C_n(X)$ Barycentric subdivision op.

For $\sigma : \Delta^n \to X$, set $m(\sigma)$ to be the smallest m s.t. $S^m(\sigma) \in C_n^{\mathcal{U}}(X)$. [Saw last time this exists because of compactness and $\mathrm{int}(A) \cup \mathrm{int}(B) = X$]

Last time we tried to define
$$\rho : C_n(X) \longrightarrow C_n^{\mathcal{U}}(X) \text{ by } \rho(\sigma) = S^{m(\sigma)}(\sigma)$$

[but this didn't give a chain map. Lets pretend I didn't notice...]

S is chain hom to ident by $P : C_n(X) \to C_{n+1}(X)$.

S^m ——— " ——— $P_m = \sum_{0 \le i \le m} P S^i$

suggests to find a map D showing ρ is chain hom to id
we take.

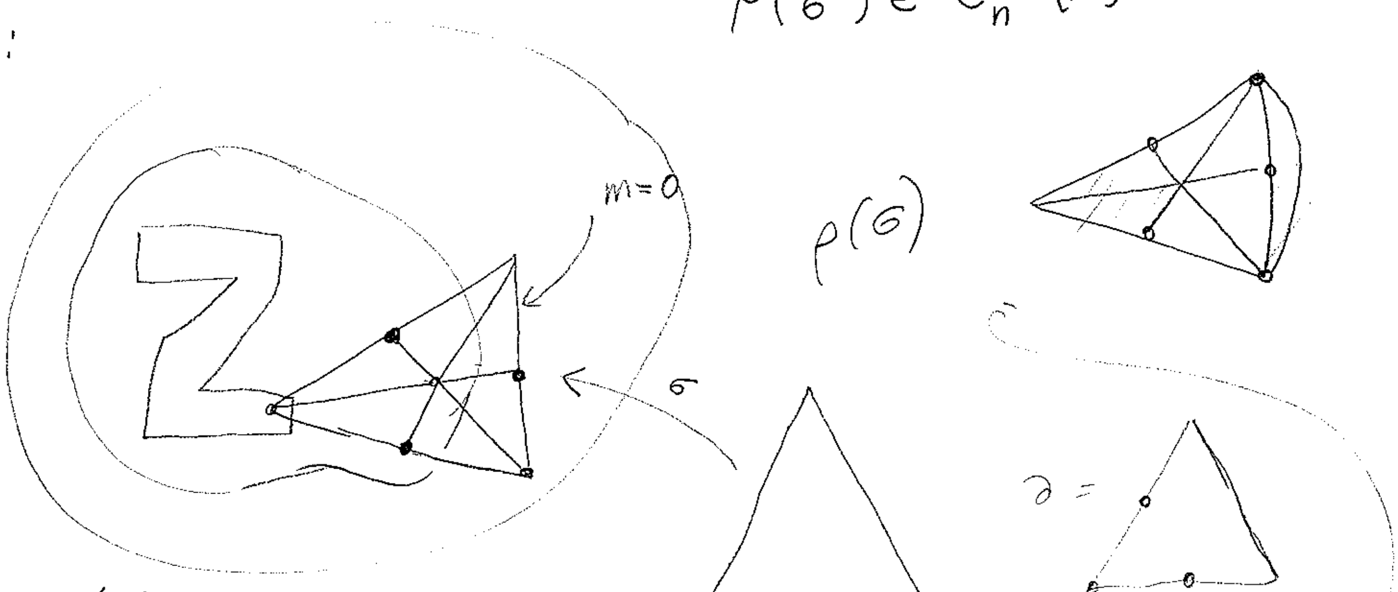
$$D\sigma = D_{m(\sigma)}\sigma. \qquad D : C_n(X) \longrightarrow C_{n+1}(X).$$

Have

$$\partial D_{m(\sigma)}\sigma + D_{m(\sigma)}\partial\sigma = \sigma - S^{m(\sigma)}\sigma \quad \left[\text{by def of chain homotopy}\right]$$

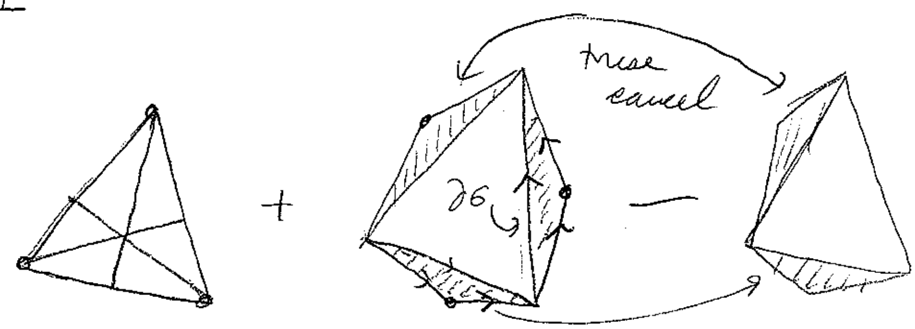
$$\Rightarrow \partial D\sigma + D\partial\sigma = \sigma - \underbrace{\left(S^{m(\sigma)}\sigma + D_{m(\sigma)}\partial\sigma - D\partial\sigma\right)}_{\rho(\sigma) \in C_n^{\mathcal{U}}(X)}$$

Ex:



$m(\sigma) = \underline{1}$

$\rho(\sigma) =$



$$S\sigma \quad + \quad D_1\partial\sigma \qquad D\partial\sigma$$

ρ is a chain map (see pic on last page, $\rho(\sigma) = \sigma - \partial D\sigma - D\partial\sigma$)

$$\Rightarrow \partial\rho(\sigma) = \partial\sigma - \partial D\partial\sigma = \rho(\partial\sigma)$$

and is chain hom to id via D.

$$C_n^{\mathcal{U}}(X) \underset{\rho}{\overset{i}{\rightleftarrows}} C_n(X)$$

$i \circ \rho \simeq \text{id}$ via D

$\rho \circ i = [?]\ \text{id}$ [exactly as $D = 0$]

End of pf of prop. ∎

<u>Pf of Excision:</u> $C_n(X) \xrightarrow{\rho} C_n^{\mathcal{U}}(X)$

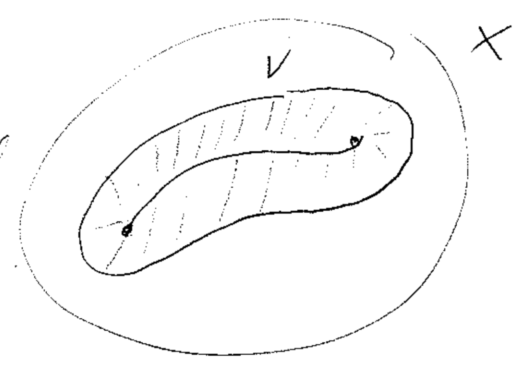
$\rho|_{C_n(A)} = \text{id}$ so this facts through to

$$\begin{tikzcd} C_n(X)/C_n(A) \arrow[r, shift left, "\rho"] \arrow[d, leftrightarrow] & C_n^{\mathcal{U}}(X)/C_n(A) \arrow[l, shift left, "i"] \arrow[d, leftrightarrow] \\ C_n(X,A) & C_n(B)/C_n(A\cap B) \end{tikzcd}$$

which is still a chain hom equiv as D also factors.

Thus $i_*: \underset{(X-Z,\ A-Z)}{H_n(B, A\cap B)} \longrightarrow H_n(X,A)$ is an isomorphism ∎

<u>Def</u>: (X, A) is a <u>good pair</u> if A is clsd and $\exists$ a nbhd V of A which def. retracts to A.



<u>Cor of Excision</u>: (X, A) a good pair. Then $q: X \to X/A$ induces an isom $H_n(X,A) \xrightarrow{q} H_n(X/A, A/A)$ for all n

$$\cong H_n(X/A) \text{ for } n \geq 1.$$

Pf: Start with last observation: $H_n(Y, pt) \cong H_n(Y)$ if $n>0$.

$$\to \underset{0}{H_n(pt)} \to H_n(Y) \xrightarrow[j_*]{\cong} H_n(Y, pt) \xrightarrow{\partial} \underset{0}{H_{n-1}(pt)} \to$$

this map is clearly injective.

$$\underset{0}{H_1(pt)} \to H_1(Y) \xrightarrow{\cong} H_1(Y, pt) \to \underset{\mathbb{Z}}{H_0(pt)} \to H_0(Y) \to \underset{H_0(Y)/\mathbb{Z}}{H_0(Y, pt)} \to 0$$

$\searrow 0 \nearrow$

<u>Claim</u>: $H_n(X, A) \xrightarrow{i_*} H_n(X, V)$ is an isom.

Pf: Long exact seq of the triple

$$\to H_n(V, A) \to H_n(X, A) \to H_n(X, V) \xrightarrow{\partial} H_{n-1}(V, A) \to$$

[why? short exact seq of chain complexes.

$$0 \to C_n(V, A) \to C_n(X, A) \to C_n(X, V) \to 0.$$

Lecture 19

Last time: Excision

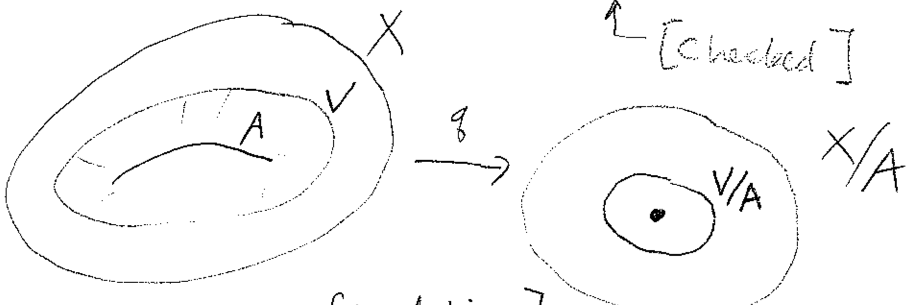
$\exists$ 

V nbhd def ret. to A clsd

Cor: (X, A) a good pair. Then $q: X \to X/A$ induces an isomorphism $H_n(X,A) \to H_n(X/A, A/A)$ for all n

$\cong H_n(X/A)$ for $n \geq 1$

[Checked]

Pf: 

[last time]

$$\begin{tikzcd}
H_n(X,A) \arrow[r, "i_*", "\cong"'] \arrow[d, "q_*"] \arrow[dr, phantom, "\circlearrowleft"] & H_n(X,V) \arrow[d, "q_*"] \arrow[dr, phantom, "\circlearrowright"] & H_n(X \setminus A, V \setminus A) \arrow[l, "i_*"', "\cong \text{ by excision}"] \arrow[d, "q_*", "\cong"'] \\
H_n(X/A, A/A) \arrow[r, "\cong"] & H_n(X/A, V/A) & H_n(X/A \setminus A/A, V/A \setminus A/A) \arrow[l, "\cong"']
\end{tikzcd}$$

[Query?] as this is a homeo

again by long exact seq of the tripple. $(V/A, A/A) \cong (A/A, A/A)$.

$\boxed{q \circ i = i \circ q}$

Commutativity now forces the leftmost q_* to be an isom. ∎

Reduced Homology $\to C_n(X) \xrightarrow{\partial_n} \cdots \to C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$

augmentation

Note $\varepsilon \circ \partial_1 = 0$ as $\partial(\sim) = \substack{+1 \\ -1} \xrightarrow{\varepsilon} 0$. $\quad (\sigma: \Delta^0 \to X) \mapsto 1$

$\tilde{H}_n(X)$ is the hom. of this complex.

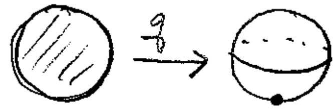
$H_n(X) = \tilde{H}_n(X)$ for $n > 0$. $H_0(X) = \tilde{H}_0(X) \oplus \mathbb{Z}$

$\tilde{H}_0(X) = \mathbb{Z}^{(\# \text{ of comp} - 1)}$

[Also MENTION, $\tilde{H}_n(X) \xrightarrow{\cong} H_n(X, pt)$.

Why consider? Now $\tilde{H}_n(X\text{-contractible}) = 0$ for all n.

Cor: (X,A) a good pair. Then $H_n(X,A) \cong \tilde{H}_n(X/A)$ $\forall n$.

Ex: $S^k = D^k/(\partial D^k \cong S^{k-1})$ 

Claim: $\tilde{H}_n(S^k) = \begin{cases} \mathbb{Z} & \text{if } k=n \\ 0 & \text{otherwise.} \end{cases}$

Base case: $n=0$ $\quad S^0 = \cdot \quad \cdot \quad \tilde{H}_0 = \mathbb{Z}$, others 0.

Induct:
$$\to \underset{\overset{\shortparallel}{0}}{\tilde{H}_n(D^k)} \to \underset{\overset{\cong}{\tilde{H}_n(S^k)}}{H_n(D^k, \partial D^k)} \xrightarrow{\cong} \underset{\overset{\cong}{S^{k-1}}}{\tilde{H}_{n-1}(\partial D^k)} \to \underset{0}{\tilde{H}_{n-1}(D^k)}$$

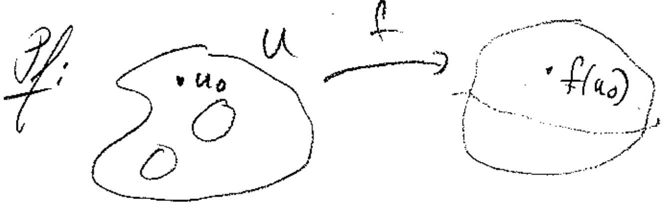
[Why reduced homology works? Still have short exact seq of chain comp.

$$\begin{array}{ccccccccc} 0 \to & C_0(A) & \longrightarrow & C_0(X) & \to & C_0(X,A) & \to 0 \\ & \downarrow \varepsilon & & \downarrow \varepsilon & & \downarrow & \\ 0 \to & \mathbb{Z} & \xrightarrow{\cong} & \mathbb{Z} & \to & 0 & \to 0 \\ & \downarrow & & \downarrow & & & \\ & 0 & & 0 & & & \end{array}$$

Cor: $\mathbb{R}^n \not\cong \mathbb{R}^m$ if $n \neq m$.

Pf: Suppose $f: \mathbb{R}^n \to \mathbb{R}^m$ is a homeo. Then $\mathbb{R}^n \setminus \{0\} \cong \mathbb{R}^m \setminus \{f(0)\}$
$$\underset{\overset{\cong}{S^n}}{\mathbb{R}^n \setminus \{0\}} \cong \underset{\overset{\cong \text{ h.e.}}{S^m}}{\mathbb{R}^m \setminus \{f(0)\}}$$

Cor [omit if running low on time].
$U \overset{\text{open}}{\subseteq} \mathbb{R}^n$ and $V^{\text{open}} \subseteq \mathbb{R}^m$. If $n \neq m$ then $U \not\cong V$.

Pf: 

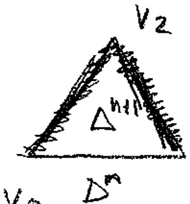
Note $H_K(U, U \setminus \{u_0\}) \overset{\text{via excision}}{\cong}$ to $H_K(\mathbb{R}^n, \mathbb{R}^n \setminus u_0) \cong \tilde{H}_K(\mathbb{R}^n \setminus u_0)$
$\cong \tilde{H}_K(S^n)$

by long exact seq of pair

[heading toward the two homologies being the same ...]

Claim: $H_n(\Delta^n, \partial\Delta^n) \cong \mathbb{Z}$ is gen by $i_n = id : \Delta^n \to \Delta^n$.

Pf: Induct on n. Clear for $n=0$ (pt, ϕ)



$\Lambda \subseteq \partial\Delta^{n+1}$ consisting of all but the 1^{st} face

$$\left[0 \to C_n(\partial\Delta^{n+1}, \Lambda) \to C_n(\Delta^{n+1}, \Lambda) \to C_n(\Delta^{n+1}, \partial\Delta^{n+1}) \to 0 \right]$$

$$\overset{\mathbb{Z}}{H_{n+1}(\Delta^{n+1}, \partial\Delta^{n+1})} \xrightarrow[\cong]{\partial} \overset{\mathbb{Z}}{H_n(\partial\Delta^{n+1}, \Lambda)} \xleftarrow{i} H_n(\Delta^n, \partial\Delta^n)$$

$$\partial\Delta^{n+1}/\Lambda = \Delta^n/\partial\Delta^n.$$

On the chain level

$$i_{n+1} \longmapsto \partial_{n+1} = \sum (-1)^j \, i_{n+1}\Big|_{j^{th}\text{ face}} = i_n \longleftrightarrow i_n \quad [i_n]$$

Thus $\partial([i_{n+1}]) = i_*([i_n])$ and so ∂ is surjective and hence an isom. So $[i_{n+1}]$ generates $H_{n+1}(\Delta^{n+1}, \partial\Delta^{n+1})$.

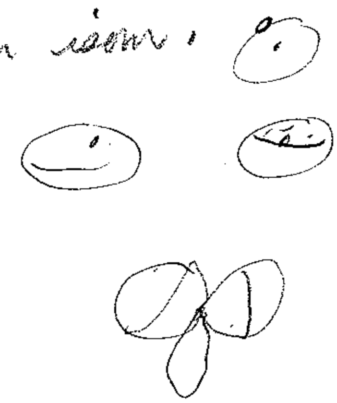
Cor: $x_\alpha \in X_\alpha$ a pt s.t. (X_α, x_α) is a good pair.

Let $Y = \bigvee_\alpha X_\alpha$ be the wedge sum at the x_α.

didn't get to.

Then if $i_\alpha : X_\alpha \to Y$ are the inclusions then

$$\bigoplus_\alpha (i_\alpha)_* : \bigoplus_\alpha \tilde{H}_n(X_\alpha) \to \tilde{H}_n(\bigvee_\alpha X_\alpha) \text{ is an isom.}$$



Pf. $\bigvee_\alpha X_\alpha = \left(\coprod X_\alpha \Big/ \coprod_\alpha \{x_\alpha\} \right)$

$H_n(\coprod_\alpha X_\alpha, \coprod_\alpha \{x_\alpha\})$

Lecture 20

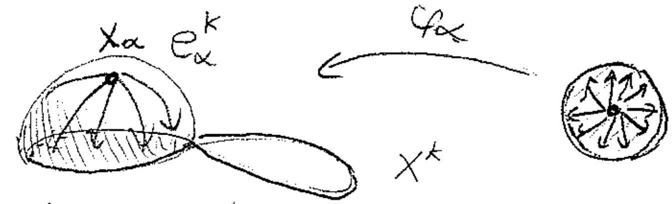
Thm: (X,A) a good pair, then $q : X \to X/A$ gives an isomorphism $H_n(X,A) \xrightarrow[q_*]{} \tilde{H}_n(X)$

Prop: $\tilde{H}_n(S^k) = \begin{cases} \mathbb{Z} & n=k \\ 0 & \text{otherwise} \end{cases}$; $H_n(\Delta^n, \partial\Delta^n) \cong \mathbb{Z}$ is gen by $i_n = id : \Delta^n \to \Delta^n$.

Today: $H_*^{\Delta}(X) \cong H_*(X)$.

Notes: 1) If X is a CW complex, and A a subcomplex, then (X,A) is good.

Ex: Suppose $X = X^k$ and $A = X^{k-1}$

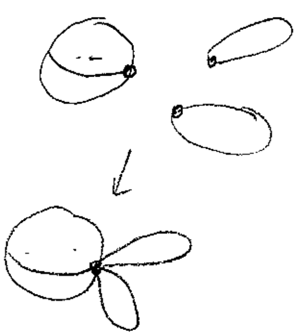


Thus $X \setminus A$ consists of k cells e_α^k. Let x_α be a pt in $\mathrm{int}(e_\alpha^k)$. Then $X \setminus \bigcup\{e_\alpha^k\}$ def retracts to A.

In general see Prop A.5.

2) Suppose x_α is a pt in X_α s.t. (X_α, x_α) is good.
Let $Y = \bigvee_\alpha X_\alpha$ along the x_α. If $i_\alpha : X_\alpha \to Y$ is the inclusion, then $\bigoplus_\alpha \tilde{H}_n(X_\alpha) \xrightarrow[\oplus_\alpha i_{\alpha *}]{} \tilde{H}_n(Y)$ is an isom.

Pf: By excision we have $H_n(\coprod_\alpha X_\alpha, \coprod\{x_n\}) \xrightarrow[\cong]{q_*} H_n(Y)$



$$\begin{array}{c} H_n(\coprod_\alpha X_\alpha, \coprod\{x_n\}) \\ \| \\ \bigoplus_\alpha H_n(X_\alpha, \{x_n\}) \\ \uparrow i_* \\ \bigoplus_\alpha \tilde{H}_n(X_n) \end{array}$$

as for any space (Z, z_0)

$$\tilde{H}_n(z_0) \to \tilde{H}_n(Z) \xrightarrow[i]{\cong} H_n(Z, z_0) \to \tilde{H}_n(z_0)$$
$$\quad 0 \qquad\qquad\qquad\qquad\qquad\qquad 0$$

X w/ Δ-complex structure $\{\sigma_\alpha : \Delta^{n_\alpha} \to X$ cell maps$\}$.

$$C_n^{\Delta}(X) \longrightarrow C_n(X) \quad [\text{a chain map}]$$

gen by σ_α

<u>Thm</u>: $H_n^{\Delta}(X) \xrightarrow[\cong]{} H_n(X)$.

<u>Pf</u>: Assume X is finite dim'l. [Full case in Hatcher].
Inductively show $H_*^{\Delta}(X^k) \cong H_*(X^k)$. Base case
$X^0 = $ pts is clear. Assume true for k. Have long exact seq

$$\begin{tikzcd}
H_{n+1}^{\Delta}(X^{k+1}, X^k) \arrow[r] \arrow[d] & H_n^{\Delta}(X^k) \arrow[r] \arrow[d, "\cong \text{ by ind}"] & H_n^{\Delta}(X^{k+1}) \arrow[r] \arrow[d] & H_n^{\Delta}(X^{k+1}, X^k) \arrow[r] \arrow[d] & H_{n-1}^{\Delta}(X^k) \arrow[d, "\cong \text{ by ind.}"] \\
H_{n+1}(X^{k+1}, X^k) \arrow[r] & H_n(X^k) \arrow[r] & H_n(X^{k+1}) \arrow[r] & H_n(X^{k+1}, X^k) \arrow[r] & H_{n-1}(X^k)
\end{tikzcd}$$

$X^{k+1}/X^k \overset{?}{=} \bigvee_\alpha S^{k+1}$ w/ one sphere for each $k+1$ cell σ_α.

$\left[\tilde{H}_n(X^{k+1}/X^k) = \begin{cases} \bigoplus_\alpha \mathbb{Z} & n = k+1 \\ 0 & n \neq k+1 \end{cases}\right]$

Only case when either $H_n^{(\Delta)}(X^{k+1}, X^k)$ is non zero is

$$H_{k+1}^{\Delta}(X^{k+1}, X^k) = \bigoplus_\alpha (\mathbb{Z}, \text{gen by } \sigma_\alpha)$$

$$\longrightarrow H_{k+1}(X^{k+1}, X^k) \cong \tilde{H}_{k+1}\Big(\bigvee_\alpha S^{k+1} = \Delta^{k+1}/\partial\Delta^{k+1}\Big)$$

$$\qquad = \bigoplus_\alpha \mathbb{Z}$$

By prop from last time, is an isom.

Now, the result follows from:

__Five lemma:__

Suppose one has a

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & D \arrow[r] \arrow[d, "\delta"'] & E \arrow[d, "\varepsilon"'] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E' \end{tikzcd}$$

commu diagram of abelian gps, with both the top and bottom rows exact. If $\alpha, \beta, \delta, \varepsilon$ are $\cong$ so is γ. [Pf: Diagram chase.].

[Recap main idea in proof. Formal prop of hom.]

[Want to understand maps as well as spaces.]

__Degree:__ $f: S^n \to S^n$ $\quad \mathbb{Z} \cong H_n(S^n) \xrightarrow{f_*} H_n(S^n) \cong \mathbb{Z}$

$$\underset{\text{gen}}{\alpha} \longmapsto (\deg f)\alpha$$

__Facts__: 1) $\deg(Id) = 1$

2) f is not onto $\Rightarrow \deg(f) = 0$. If p is the missed point then $H_n(S^n) \xrightarrow{f_*} H_n(S^n \setminus p) \xrightarrow{i_*} H_n(S^n)$

$$\alpha \longmapsto 0 \longmapsto (\deg f)\alpha.$$

zero group as $\cong \mathbb{R}^n$

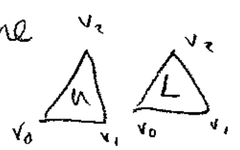
3) $f \simeq g \Rightarrow \deg f = \deg g$. [in fact, the converse is true, will show in 151b.]

4) all $n \in \mathbb{Z}$ are degrees of some map. [Think of the case of S^1]

5) Suppose f is a reflection in a plane through 0. Then $\deg f = -1$.

$S^n \subseteq \mathbb{R}^{n+1}$

S^n has Δ-complex structure w/ 2 n-simplices.



$(x_1, \dots, x_{n+1}) \overset{f}{\longmapsto} (x_1, \dots, x_n, -x_{n+1})$

$H_n(S^n)$ is gen by $[U - L]$

$$\overset{f_*}{\longmapsto} [L - U] = -[U - L]$$

6) Antipodal Map: $A: S^n \to S^n \quad A(x) = -x$

$A = (\text{comp of } n \text{ refl.}) \Rightarrow \deg A = (-1)^{n+1}$.

$\deg(f \circ g) = (\deg f)(\deg g)$. $\quad H_n(S^n) \longrightarrow H_n(S^n) \longrightarrow H_n(S^n)$

$\alpha \longmapsto (\deg f)\alpha \longmapsto (\deg f)\deg(g)\alpha$.

Lecture 21

<u>HW #8 Due Nov 17:</u> See 2.1 #26

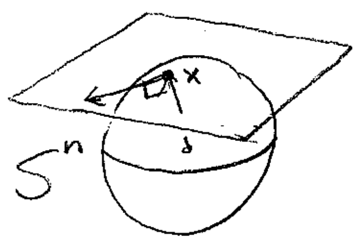
See 2.2 # 2, 4, 8, 11, 13

<u>Last time:</u> $f: S^n \to S^n \quad H_n(S^n) \longrightarrow H_n(S^n)$

$\alpha \longmapsto (\deg f)\alpha$

$A(x) = -x$, $\deg = (-1)^n$

$T_x S^n = \{ v \in \mathbb{R}^{n+1} \mid x \cdot v = 0 \}$

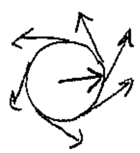


<u>Vector fields:</u> S^n $\quad V: S^n \to \mathbb{R}^{n+1}$

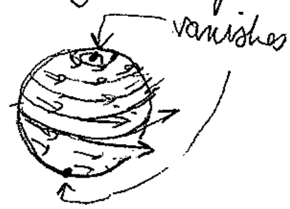
$V(x) \in T_x S^n$

<u>Thm</u> S^n has a nowhere vanishing vector field $\Leftrightarrow$ n is odd

<u>$n=1$:</u>

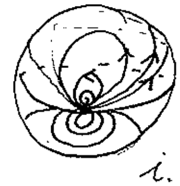


<u>$n=2$:</u>

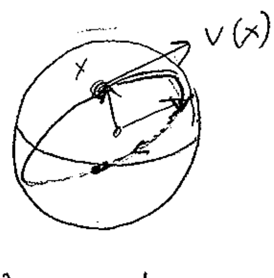


[Query: can we have only one pt where it vanishes?]

Yes:



<u>Pf:</u> ($\Rightarrow$). Suppose v is such a vector field. Rescale v so it has unit length, i.e. $\dfrac{V(x)}{|V(x)|}$



$f_t(x) = \cos(\pi t)\, x + \sin(\pi t)\, v(x)$

$f_t: S^n \times [0,1] \to S^n \quad f_0 = id \quad f_1 = A$.

So $A \simeq id \Rightarrow 1 = \deg(id) = \deg(A) = (-1)^{n+1}$.

$\Leftarrow \; V(x_1, \ldots, x_{n+1}) = (-x_2, x_1, -x_4, x_3, \ldots, -x_{n+1}, x_n)$

Q: What spaces is S^n the univ. cover for? [necess. manifolds]

Ex: [Q] $\mathbb{R}P^2 = S^2/_{x \sim -x}$ 

$\mathbb{R}P^n = S^n/_{x \sim -x}$.

Ex: Lens spaces: $n \geq 0$, $a \in [1, n-1]$ rel. prime to n.

$\zeta_n = e^{2\pi i/n}$ $\quad \mathbb{Z}/n$ acts on $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$

via $(z_1, z_2) \longmapsto (\zeta_n z_1, \zeta_n^a z_2)$.

$S^3 \longrightarrow S^3/\mathbb{Z}/n = L_{a/n}$ is a covering map. [Q: Why?] [A: free, finite]

$\pi_1(L_{a/n}) = \mathbb{Z}/n$, Saw $L_{1/n}$ on HW.

Fact: $L_{1/7}$ and $L_{2/7}$ are 3-manifolds which are homotopy equivalent, but not homeomorphic.

[Lens spaces exist in any odd dim.]

free is important! Q: Example.

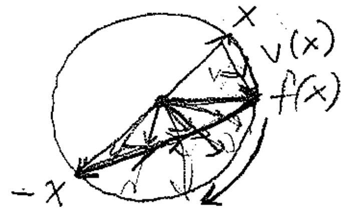
Thm: If n is even, the only finite grp which can act freely on S^n is $\mathbb{Z}/2$. [$\mathbb{Z}/2$ does act via the antipodal map]

Pf: Suppose G acts freely on S^n. Consider $d: G \to \mathbb{Z}$ given by $d(g) = \deg(g)$. Now $d(g) = \pm 1$ as g is a homeo, and [as $\deg(f \circ g) = \deg(f)\deg(g)$] d is a homo. $G \to \{\pm 1\} \cong \mathbb{Z}/2$

Claim: If $S^n \circlearrowleft f$ has no fixed pts, then $f \simeq A$.

Pf of thm from claim: if n is even, every $g \neq 1$ in G has degree -1. Thus d has no kernel, and $G = \mathbb{Z}/2$.

Pf of Claim:



$$f_t(x) = \frac{(1-t)(-x) + t f(x)}{\| (1-t)(-x) + t f(x) \|}$$

$\square$

<u>Q</u>: $f: S^n \circlearrowleft$ with no fixed pts, $f^2 = 1$ $\exists$? $g: S^n \circlearrowleft$ s.t. $g^{-1} \circ f \circ g = A$.

$\iff S^n/f \cong \mathbb{RP}^n$

<u>$n=1$</u>: yes. <u>$n=2$</u>: Yes by class of surfaces. <u>$n=3$</u> yes: Livesay 1960.

<u>$n \geq 4$</u>: no.

<u>Thm</u> (Perelman 2003): $S^3/_{\mathbb{Z}/n}$ is always a Lens space.

He also proved:

<u>Poincaré Conj</u>: M^3 cpt 3-mfld [w/o ∂]. If $\pi_1 M = 1$ then $M^3 \cong S^3$.

[Proposed in 1900 or so. Note that is false

~~Conj~~: M^3 a noncpt 3-mfld which is contractible. Then $M^3 \cong \mathbb{R}^3$.

[via the example from the exam.]

[If time permits blather about Poincaré in high dims.]

Lecture 22 Last time: Applications of degree:

$$f: S^n \to S^n$$

$$\mathbb{Z} = H_n(S^n) \longrightarrow H_n(S^n) = \mathbb{Z}$$

$$\alpha \longmapsto (\deg f)\alpha$$

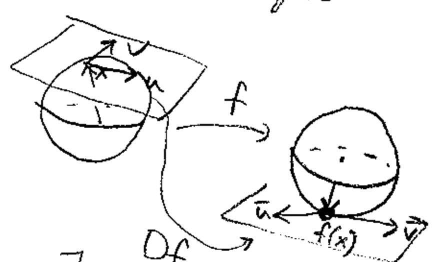
Today: What degree measures.

Next time: CW homology.

[How to comp deg in general? What does it really measure?]

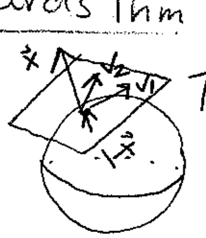
$f: S^n \to S^n$ smooth [hand wave def, f extends to some open nbhd of S^n as a smooth function. Motivate degree.]

For $x \in S^n$, get $Df: T_x S^n \to T_{f(x)} S^n$ a linear map



Def: y is a regular value if $Df: T_x S^n \to T_y S^n$ is an isom for all $x \in f^{-1}(y)$. [$\Rightarrow$ local difeo near x]

Sard's Thm The set of all regular values is dense. [of full Lebesgue measure!]



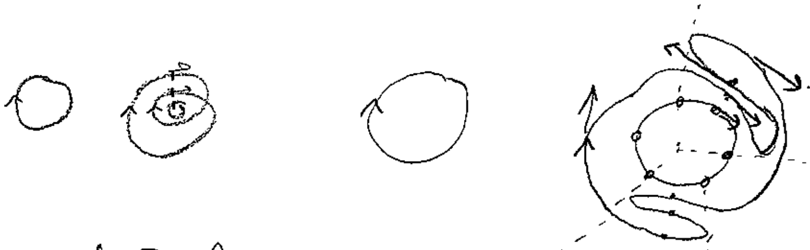
$T_x S^n$ has an orientation induced by $\det\begin{pmatrix} | & & | & | \\ v_1 & \cdots & v_n & x \\ | & & | & | \end{pmatrix} > 0$.

If $Df: T_x S^n \longrightarrow T_{f(x)} S^n$ is an isom set

$$\varepsilon(x) = \begin{cases} +1 & \text{if } Df \text{ pres orient} \\ -1 & \text{otherwise} \end{cases}$$

Thm: $f: S^n \to S^n$ a smooth map, $y \in S^n$ a regular value.
Then $\deg f = \sum_{x \in f^{-1}(y)} \varepsilon(x)$.

Ex $n = 1$ $\quad z \longmapsto z^n$

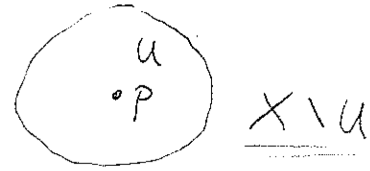


Cor: $\# f^{-1}(y) \equiv (\deg f) \bmod 2$, for y a regular value.

[Want to define ε for any map, or pt.]

Local Homology: $p \in X$ $\quad H_n(X, X \setminus p)$

$p \in U^{\text{open}}$ $\quad \cong \; H_n(U, U \setminus p)$ $\quad$ [assuming p is closed]



If $U \cong \mathbb{R}^k$: $\quad \underset{0}{\tilde{H}_n(U)} \to H_n(U, U\setminus p) \to \tilde{H}_{n-1}(\underbrace{U \setminus p}_{\cong_{h.e.} S^{k-1}}) \to \underset{0}{\tilde{H}_{n-1}(U)}$

$\Rightarrow H_n(U, U\setminus p) = \begin{cases} \mathbb{Z} & n = k \\ 0 & \text{otherwise} \end{cases}$

Cor: M an m-manifold, N an n-manifold. If $m \neq n$ then $M \not\cong N$.

[Pf: a homeo induces an isom on local homology.]

—— o ——

Local Degree: $f: S^n \to S^n$ $\quad \underset{\mathbb{Z}}{H_n(S^n, S^n \setminus x)} \xrightarrow{f_*} \underset{\mathbb{Z}}{H_n(S^n, S^n \setminus f(x))}$

$\bar{\alpha} \longmapsto (\deg_x f)\, \bar{\alpha}$

where $\bar{\alpha}$ is the gen coming from

$H_n(S^n) \to H_n(S^n, S^n \setminus x)$

$\alpha \longmapsto \bar{\alpha}$

$\boxed{\text{This def is in error: what if } \cancel{S^n \setminus x} \; f^{-1}(x) \cap S^n \setminus \{x\} \neq \emptyset}$

Thm: $f: S^n \to S^n$. If $f^{-1}(y)$ is finite, then $\deg f = \sum_{x \in f^{-1}(y)} \deg_x f$

If f is smooth and Df_x is an isom, then $\deg_x f = \varepsilon(x)$.

[Note f is a homeo near x so $\deg_x f = \pm 1$, choose chart so locally homotopic to its derivative.

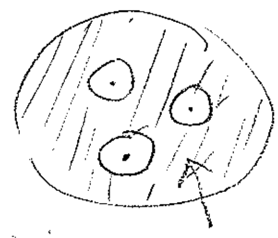
Key: $A: \mathbb{R}^n \to \mathbb{R}^n$ a linear isom. Then $\deg_0 A = \operatorname{sign}(\det A)$.

[Should have assigned on HW??]

Pf: $f^{-1}(y) = \{x_i\}$, U_i = disjoint nbhds of x_i:

$$\begin{tikzcd}
\bigoplus H_n(U_i, U_i \setminus x_i) \arrow[r, "i", "\cong"'] & H_n(S^n, S^n \setminus f^{-1}(y)) \arrow[r, "f_*"] & H_n(S^n, S^n \setminus y) & \bar{\alpha} \\
\alpha_i & H_n(S^n) \arrow[u, "j"] \arrow[r, "f_*"'] & H_n(S^n) \arrow[u, "j"] & \alpha \arrow[u, mapsto] \\
& \alpha & &
\end{tikzcd}$$

[from long exact seq of pair]



$H_n(S^n, S^n \setminus f^{-1}(y)) \cong H_n(\coprod U_i, \coprod (U_i \setminus x_i))$ by excising $Z = S^n \setminus \coprod U_i$

$$\begin{array}{c} \cong \\ \bigoplus_i \left\{ \begin{array}{ll} H_n(U_i, U_i \setminus x_i) & \text{gen by } \alpha_i \\ \downarrow i & \quad \downarrow \\ H_n(S^n, S^n \setminus x_i) & \text{gen by } \bar{\alpha} \end{array} \right. \end{array}$$

Note: $i^{-1} \circ j(\alpha) = \sum \alpha_i$

What is $f_* \circ i(\alpha_i)$? A: $(\deg_{x_i} f)\bar{\alpha}$

$$\begin{tikzcd}
\alpha_i & H_n(U_i, U_i \setminus x_i) \arrow[r, "f_*"] \arrow[d, "\cong"] & H_n(S^n, S^n \setminus y) \\
\bar{\alpha} \arrow[u] & H_n(S^n, S^n \setminus x_i) \arrow[ur, "f_*"'] & (\deg_{x_i} f)\bar{\alpha}. \\
& \bar{\alpha} \arrow[ur, mapsto] &
\end{tikzcd}$$

So: $\sum \alpha_i \to \sum (\deg_{x_i} f)\bar{\alpha}$

$\Rightarrow \deg f = \sum (\deg_{x_i} f)$ ✓.

Ex: Suspension of $z \mapsto z^n$ has degree n on any S^k.

Ex: Consider $S^k \longrightarrow \bigvee_{i=1}^{m} S^k \longrightarrow S^k$ map has degree m.