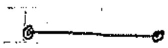
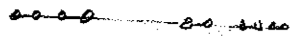
Ma 151a: Algebraic and Geometric Topology

Caltech, Fall 2006

Topological Spaces: X set w/ collection of open sets

pt,  $I=[0,1]$,  Cantor Set, , , I^∞, ...

[Basic examples from point-set topology.]

[Functional Analysis: spaces of cont, smooth fns (very large)]

[$\mathbb{R}^n$, manifolds,   , S^3]

Geometric Topology

Problem: X, Y top spaces. Are they [the same] homeomorphic.

[Fundamental] **Yes:** Produce homeo (doughnut = coffee cup [joke])

No: Give prop. that distinguishes them.

Easy: $(0,1) \not\cong [0,1]$ [Query: Compactness]

$\mathbb{R} \not\cong \mathbb{R}^2$ [Query: connectedness of space $\setminus$ pt]

Harder: $\mathbb{R}^2 \not\cong \mathbb{R}^3$,  $\not\cong$ .

[Need additional properties / invariants. An imp. source of such is]

<u>Algebraic Topology</u>:

$$X \rightsquigarrow F(X)$$ — alg object, e.g. abelian group [or alg.] (easier to tell apart)

[top space]

[only dep on homeo type of X]

Also

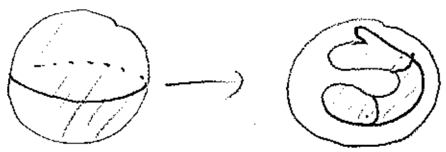
$X \xrightarrow{f} Y$ get $F(X) \xrightarrow{f_*} F(Y)$

cont. map

[resp. alg. str, e.g. is a homomorphism.]

<u>Brouwer Fixed Point Thm</u>:

Let $f: D^n \to D^n$ be cont, where $D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ [The closed unit disc] Then $\exists x \in D^n$ w/ $f(x) = x$.



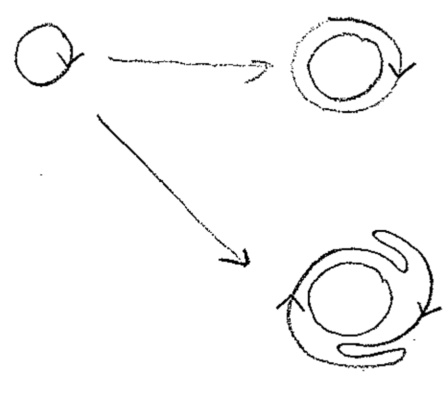
[In 151 this year, the focus will be on algebraic top, esp of the objects of geometric.]

<u>Main Invariants of Alg Top</u>:

<u>Fundamental Group</u>: $\pi_1 X$ = "the set of loops $S^1 \xrightarrow{\gamma} X$, up to cont deformation"

[has a group str.]

$$\pi_1(\bigcirc) = \mathbb{Z}$$



same element of π_1!

[Great, but limited.]

<u>Higher Homotopy groups</u>: $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| \leq 1\}$

 , S^3, ...

$\pi_n(X) = S^n \longrightarrow X$ [up to deformation.]

abelian gp, very powerful and capture a huge ammount of info about X. Really hard to compute! $\pi_n(S^2 =$ $)$ are not all known.

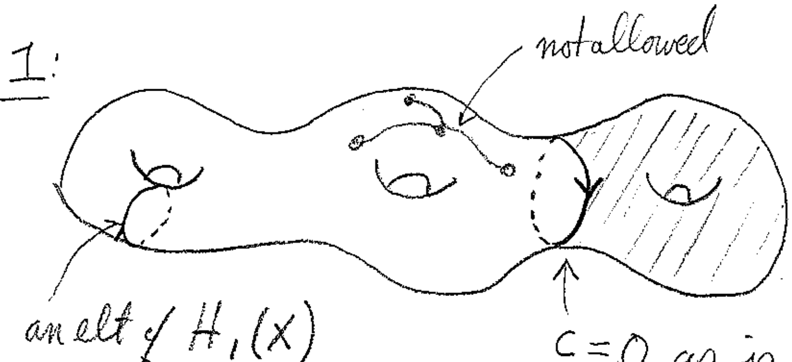
[Next quarter w/ Danny Calegari]

<u>Homology</u> [bulk of this qtr]

$H_n(X) =$ "n-dim'l things w/o boundaries / boundries of n+1 dimension things"

[an abelian group]

<u>n = 1</u>:



[Easy compute, but still very useful]

<u>Cohomology</u>: $H^n(X)$ – "dual" notion to homology

[When X is a smooth manifold, $H^n(X)$ can be const. from diff. forms.]

$H^n(X) \times H^j(X) \longrightarrow H^{k+j}(X)$ a product

making $\bigoplus H^n(X)$ an algebra. [Winter quarter.]

[While we will be covering these from a top. viewpt, (co) homology is an important part of many areas of math.]

151c: char. classes. Daniel Groves

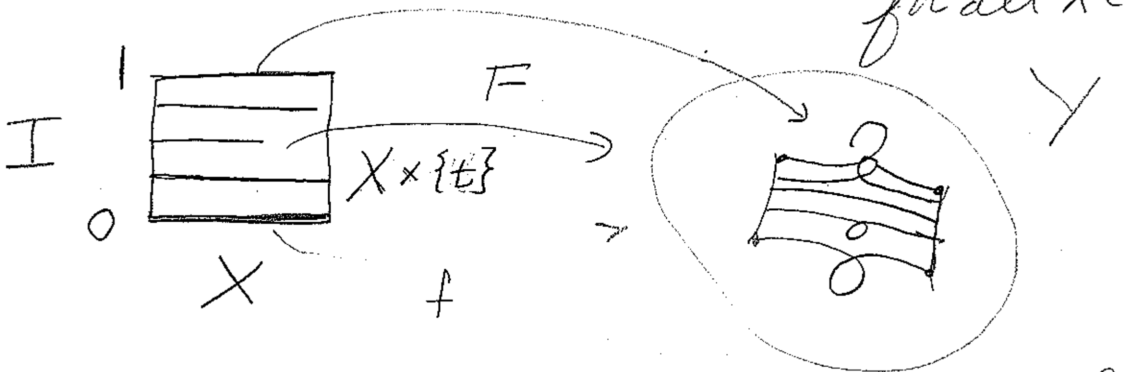
Go over syllabus.

Lecture 2

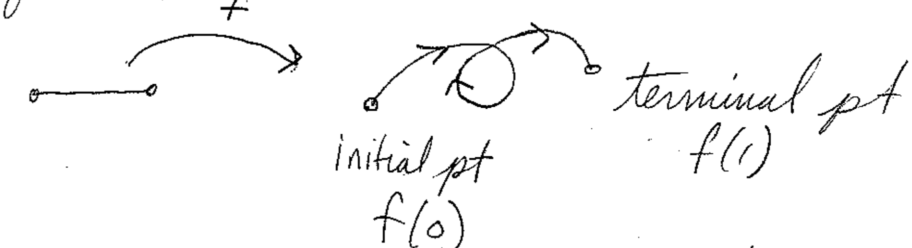
Reminder: HW Due Friday

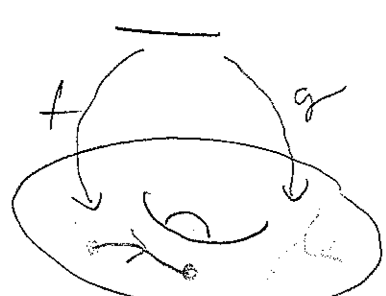
Today: Def of fund. group + covering space [maps of S^1 into space up to cont def]

Def: Cont. maps $f, g: X \to Y$ are homotopic ($f \simeq g$) if $\exists$ a cont map $F: X \times I \to Y$ s.t. $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for all $x \in X$.
($I = [0,1]$)



Def: A path Y is just a cont map $f: I \to Y$





Query: Are these two path homotopic?

[If Y is path connected, then any two paths are homotopic, so need to refine the definition.]

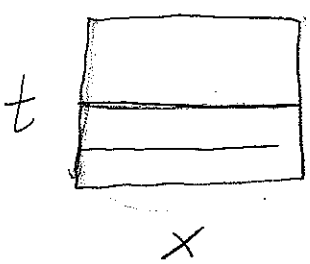
Def: Paths $f, g : I \to X$ going from y_0 to y_1 are **path homotopic** ($f \simeq_p g$) if $\exists\, F : I \times I \to Y$

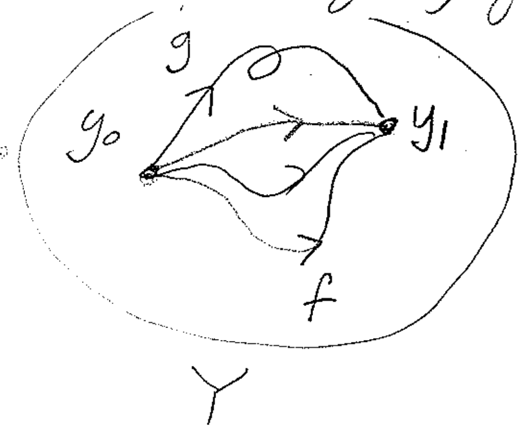
s.t. $F(x,0) = f(x) \quad \forall x \in I$

$F(x,1) = g(x)$

and $F(0,t) = y_0$

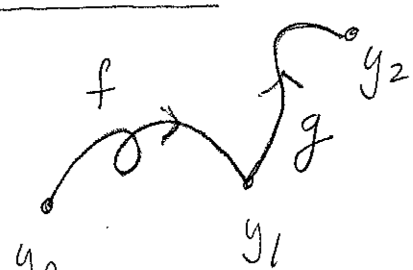
$F(1,t) = y_1$





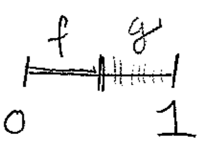
[Remember, I'm trying to create a group out of loops/paths...]

Concatenation: f a path from y_0 to y_1, g path from y_1 to y_2



Let $f \cdot g$ = composit path from y_0 to y_2

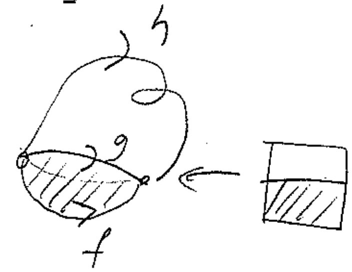
$$f \cdot g(s) = \begin{cases} g(2s-1) & s \in [1/2, 1] \\ f(2s) & s \in [0, 1/2] \end{cases}$$



[To cut down of the # of paths]

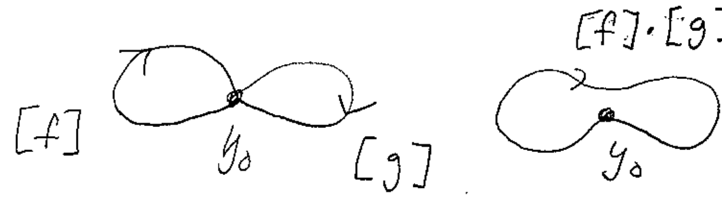
Note: $\simeq_p$ is an equivalence relation.

$[f]$ – path homotopy class of f.



$[f] \cdot [g] = [f \cdot g]$ (can check is well defined, etc.)

Def: Y top space, $y_0 \in Y$. The *fundamental group* of Y based at y_0 is $\pi_1(Y, y_0) = \left\{ [f] \mid f \text{ a path starting and ending at } y_0 \right\}$ together with the op $\cdot$.



Check [group axioms]

Assoc: $\left([f]\cdot[g]\right)\cdot[h] = [f]\cdot\left([g]\cdot[h]\right)$

Ident: $1 = [\text{const path at } y_0]$

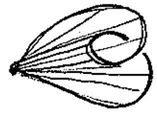
Inverses: $[f]^{-1} = [\bar{f}]$ where $\bar{f}(s) = f(1-s)$ [i.e. f backwards]

[If time permits, explain one on a class note, else just refer to Sec 1.1 of Hatcher.]

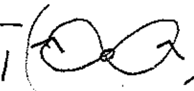
Prop: If y_0 and y_1 can be joined by a path, then
$\pi_1(Y, y_0) \cong \pi_1(Y, y_1)$

$[f] \mapsto [\bar{g}\cdot f \cdot g]$

Ex: $\pi_1 \mathbb{R}^n = 1$



$\pi_1 S^1 = \mathbb{Z}$

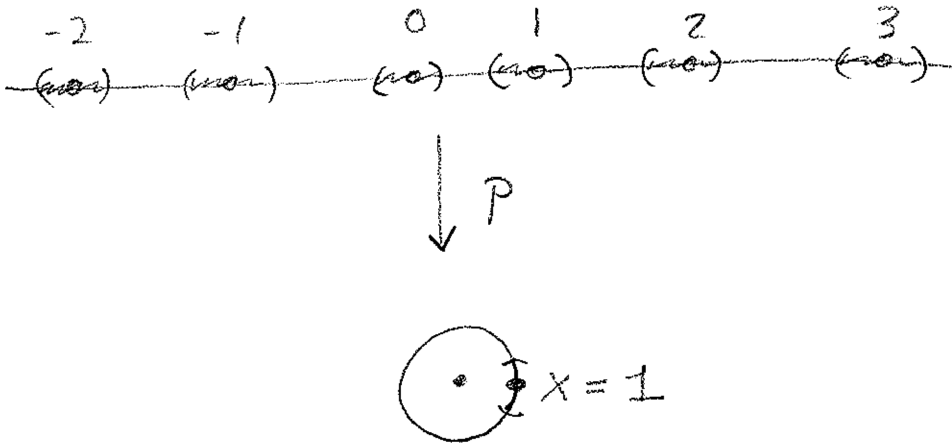
$\pi_1($  $) =$ Free gp of two gen

$\pi_1($  $) = \langle a,b,c,d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1} = 1 \rangle$

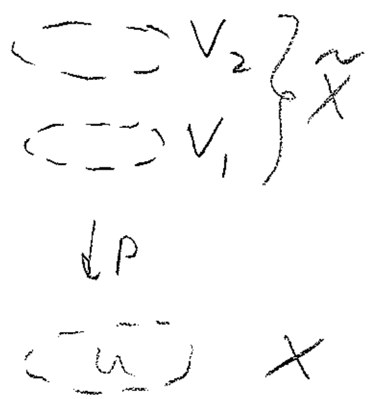
[Of course, I haven't justified any of this, we need tools, namely covering spaces which also help us understand π_1.]

Covering Spaces:

Ex: $p: \mathbb{R} \to S^1$ $\quad p(t) = (\cos 2\pi t, -\sin 2\pi t) = e^{-2\pi t i}$

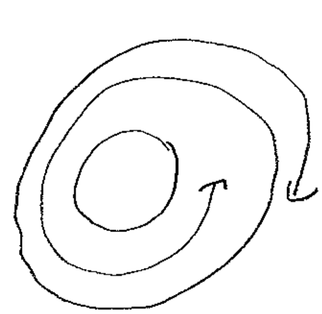


Def: Let $p: \tilde{X} \to X$ be a cont map.
$U^{open} \subseteq X$ is evenly covered if $p^{-1}(U) =$ disjoint union $V_\alpha^{open} \subseteq \tilde{X}$
where $p|_{V_\alpha}: V_\alpha \to U$ is a homeo for each α.

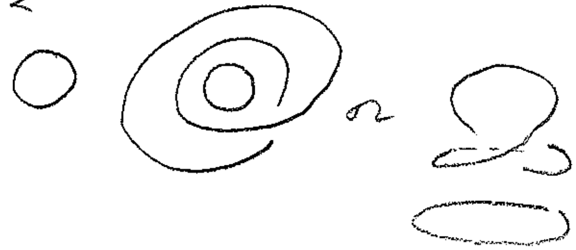


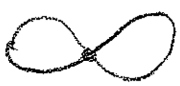
Def: $p: \tilde{X} \to X$ is a __covering map__ if every $x \in X$ has a nbhd U which is evenly covered. $\tilde{X}$ is called a covering space of X.

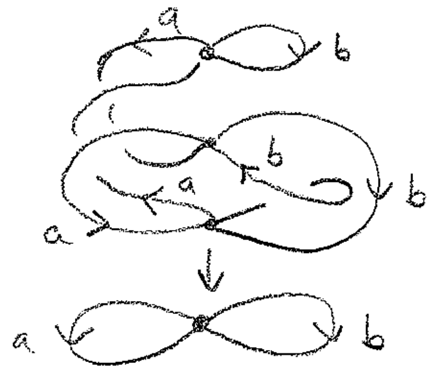
Note: Not enough that $p: \tilde{X} \to X$ is a local homeo. e.g. $p|_{(0,2)}$

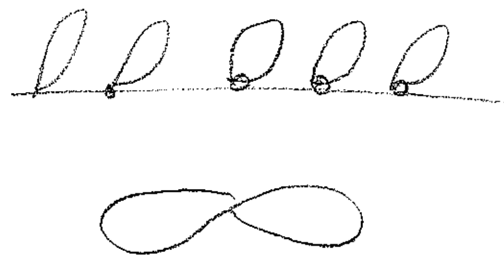


Further examples: $p: S^1 \longrightarrow S^1$ $\quad n=2$
$z \longmapsto z^n$



$X =$ 

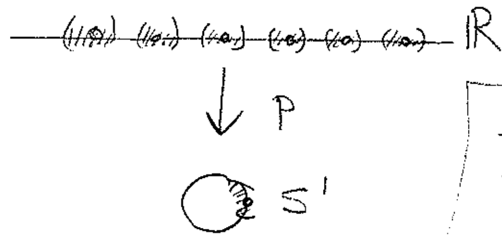




see pg 58 for many examples.

Lecture 3 HW #2 Due Fri Oct 6: §1.1 #7, 10 §1.3, # 6, 7, 9, 15

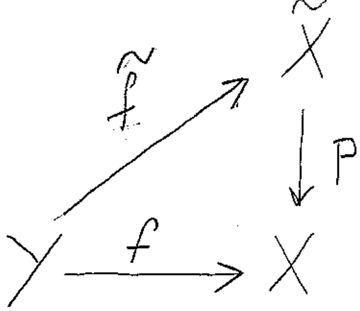
Last time: $p: \tilde{X} \to X$ is a covering map if every $x \in X$ has an evenly covered open nbhd U, i.e. $p^{-1}(U) = \coprod_\alpha V_\alpha$ open s.t. $p|_{V_\alpha}$ is a homeo.

Ex: 

$p(t) = e^{-2\pi i t}$

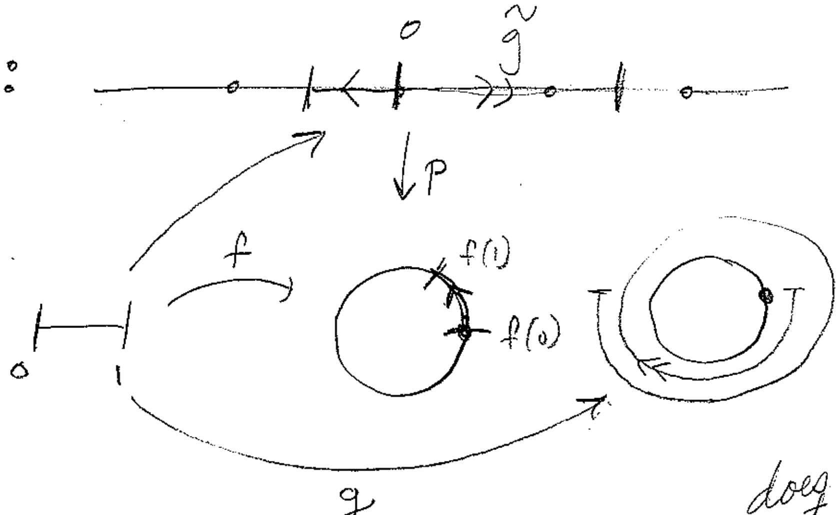
Today: Lifting maps to covering spaces, and $\pi_1 S^1$!

Def: Let $p: \tilde{X} \to X$, $f: Y \to X$ be maps

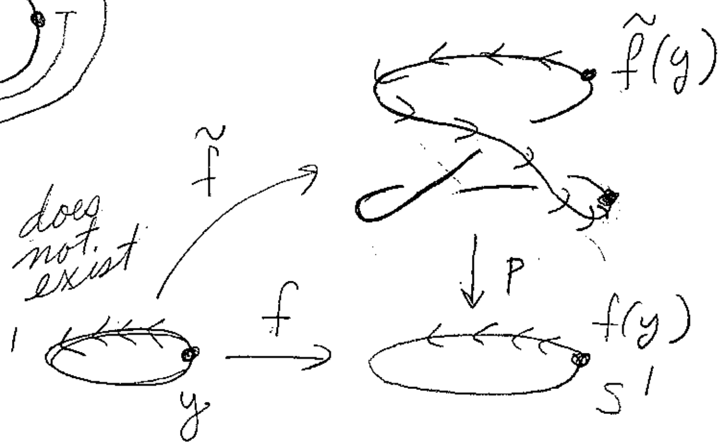


A lift of f is a map $\tilde{f}: Y \to \tilde{X}$ s.t. $p \circ \tilde{f} = f$.

Ex:



Non Ex: $p: S^1 \longrightarrow S^1$, $z \longmapsto z^2$; $f: S^1 \xrightarrow{id} S^1$



[The fundamental group will tell use exactly when a lift exists, will be able to compute π_1.]

[note non-uniqueness of lifts in 1st example.]

Thm: $p: \tilde{X} \to X$ a covering map, $f: I \to X$ a path starting at x_0. For each $\tilde{x}_0 \in p^{-1}(x_0)$, $\exists$ a unique lift of f to a path $\tilde{f}$ starting at $\tilde{x}_0$.

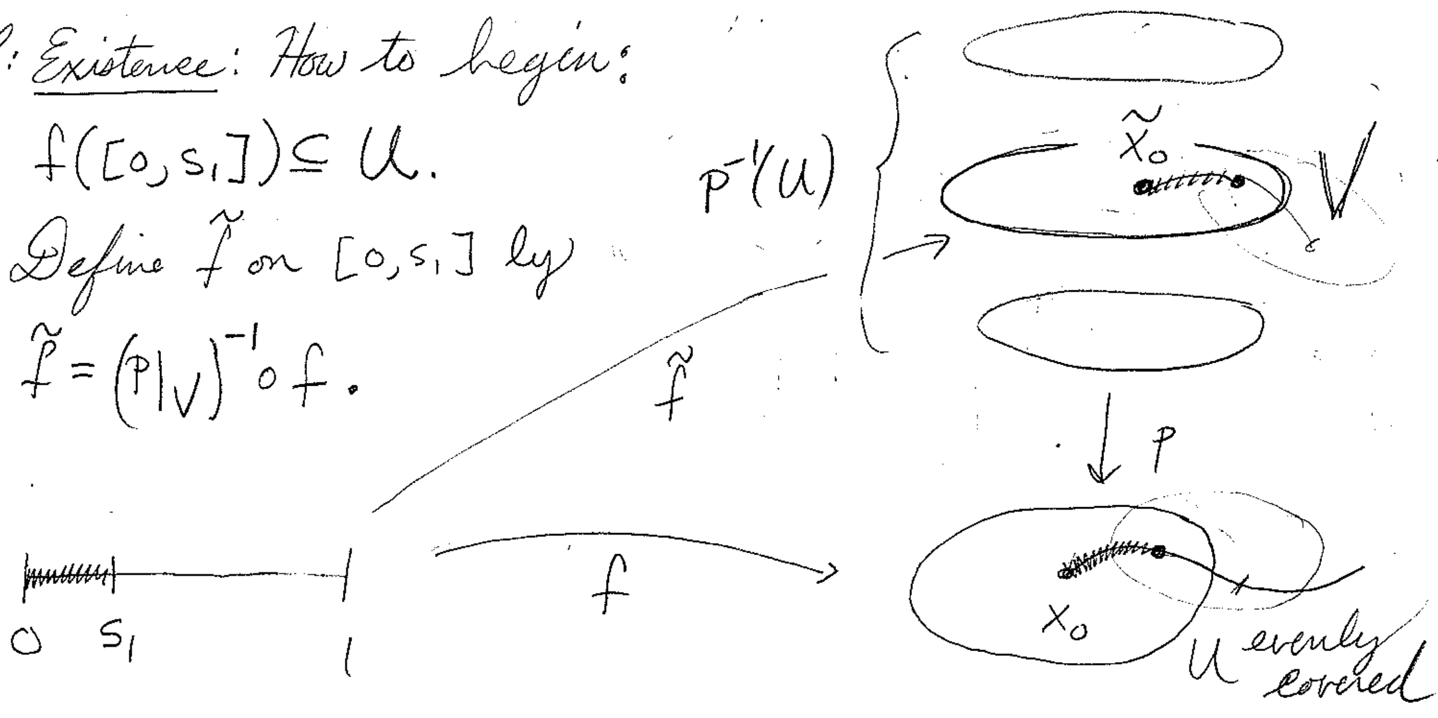
$$\begin{array}{ccc} & & \tilde{X} \\ & \overset{\tilde{f}}{\nearrow} & \downarrow p \\ I & \xrightarrow{f} & X \end{array}$$

Pf: Existence: How to begin:

$f([0, s_1]) \subseteq U$.

Define $\tilde{f}$ on $[0, s_1]$ by

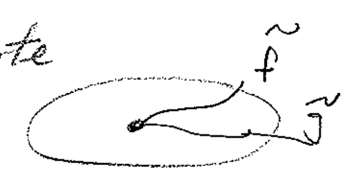
$\tilde{f} = (p|_V)^{-1} \circ f$.



Now repeat this process noting that compactness gives a finite partition $0 = s_0 < s_1 < \dots < s_n = 1$ where each $f([s_i, s_{i+1}]) \subseteq U_i$ evenly covered.

Got confused here. What if V is not conn?

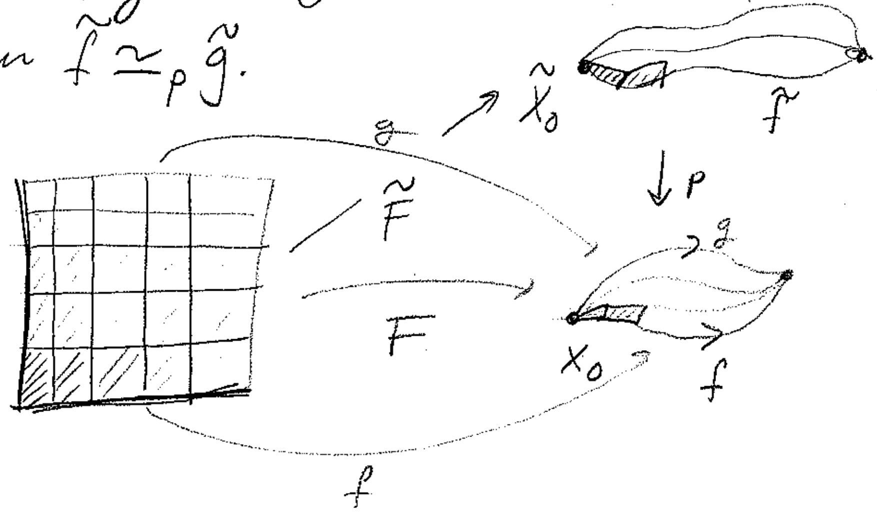
Uniqueness: If $g : I \to \tilde{X}$ we some other lift, note by connectedness $g([0,s_1])$ lies in V. Must have $\tilde{f} = g$ on $[0,s_1]$ as $p|_V$ is a bijection.



∎

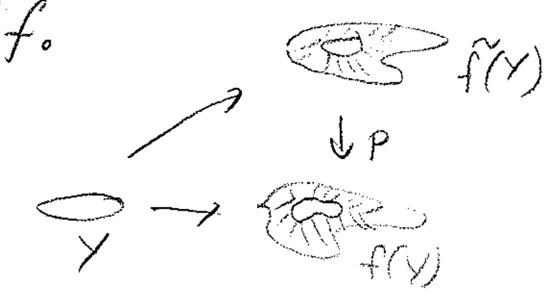
Addendum: Suppose $f \simeq_p g$ and $\tilde{g}$ is the lift of g starting at $\tilde{x}_0$. Then $\tilde{f} \simeq_p \tilde{g}$.

Pf:



Prop 1.30: $p : \tilde{X} \to X$ covering map, $f : Y \to X$ a map $F : Y \times I \to X$ a homotopy of f. If f lifts to $\tilde{f}$ then $\exists$ a unique lift of F to a homotopy of $\tilde{f}$.

[Thm was $Y = pt$, addenum was $Y = I$]

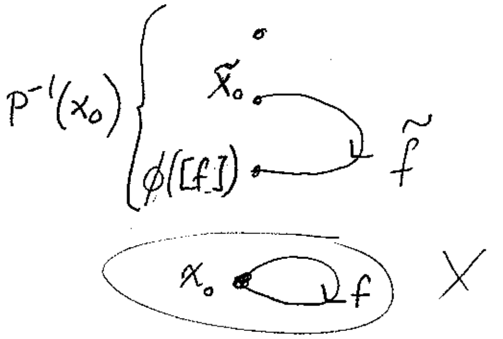


[Now lets bring π_1 directly into the picture]

Def: $p:(\tilde{X},\tilde{x}_0) \to (X, x_0)$ a covering map

[Describe notation $\tilde{}$] The lifting correspondence

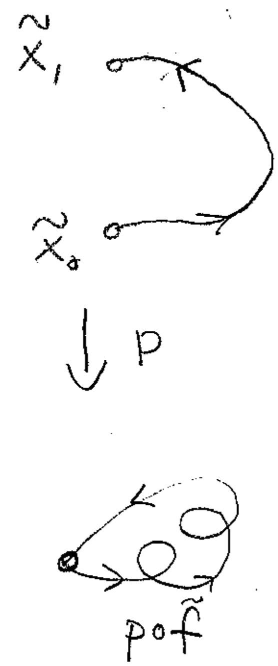
$\phi : \pi_1(X,x_0) \longrightarrow p^{-1}(x_0)$ is $\phi([f]) =$ end pt of the lift $\tilde{f}$ of f based at $\tilde{x}_0$



Def: Z is simply connected if it is path connected and $\pi_1 Z = 1$

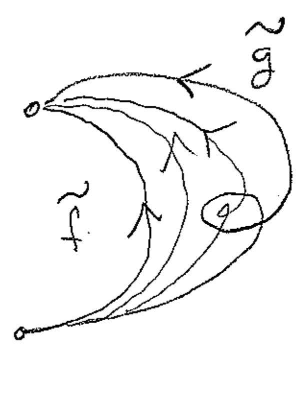
[Ex: $\mathbb{R}^n$]

Thm: $p:\tilde{X} \to X$ a covering space w/ $\tilde{X}$ simply conn Then ϕ is a bijection.



Pf: onto: $\tilde{x}_1 \in p^{-1}(x_0)$ As $\tilde{X}$ is path conn, $\exists$ a path $\tilde{f}$ from $\tilde{x}_0$ to $\tilde{x}_1$. Then $[p\circ\tilde{f}] \in \pi_1(X,x_0)$ and $\phi([p\circ\tilde{f}]) = \tilde{x}_1$.

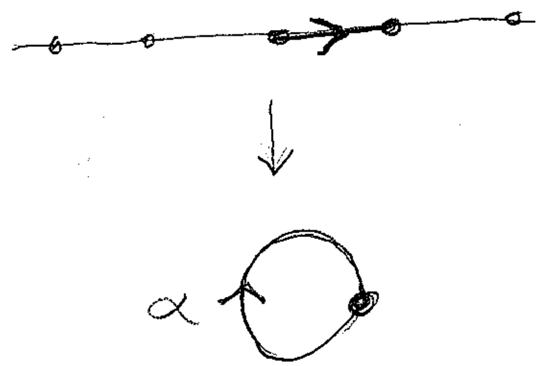
1-1: Suppose $\phi([f]) = \phi([g])$. As $\pi_1\tilde{X} = 1$, we have $\tilde{f} \simeq_p \tilde{g}$ via some homotopy $\tilde{F}$.



$\left([\tilde{f}\cdot\bar{\tilde{g}}] = [\text{const } \tilde{x}_0] \text{ hence } [\tilde{f}] = \underbrace{[\tilde{f}]\cdot[\bar{\tilde{g}}]}_{=1}\cdot[\tilde{g}] = [\tilde{g}] \right)$

Then $p\circ F$ is a path homotopy from f to g, so $[f]=[g]$.

Thm: $\pi_1(S^1, 1) = \mathbb{Z}$



Pf: Consider the cover $p: \mathbb{R} \to S^1$

By thm, we have a bijection

$$\pi_1(S^1, 1) \xrightarrow{\phi} \mathbb{Z}.$$

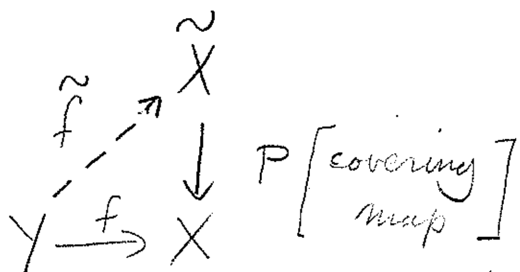
Claim: ϕ is a homomorphism

Note $\phi([\alpha^n]) = n$, so as ϕ is a bijection, we see that $\pi_1(S^1)$ is cyclic, generated by $[\alpha]$.

Lecture 4

Last time: Lift

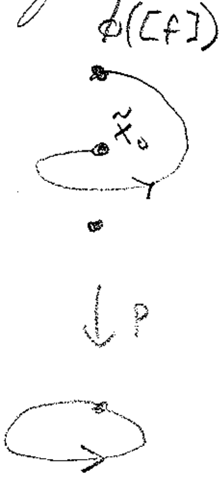
$f = p \circ \tilde{f}$



Things that lift: 1) paths 2) homotopies of things that lift.

Lifting correspondence $(\tilde{X}, \tilde{x}_0) \xrightarrow{p} (X, x_0)$ a covering map.

$$\phi: \pi_1(X, x_0) \to p^{-1}(x_0) \text{ via } \phi([f]) = \text{end pt of lift of } f \text{ starting at } \tilde{x}_0$$



[Used to compute $\pi_1 S^1$]

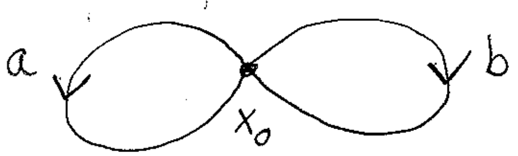
Today: • induced homomorphism on π_1.

• how π_1 solves the lifting problem

[• start to see conn between cover and corr. subgroup of π_1]

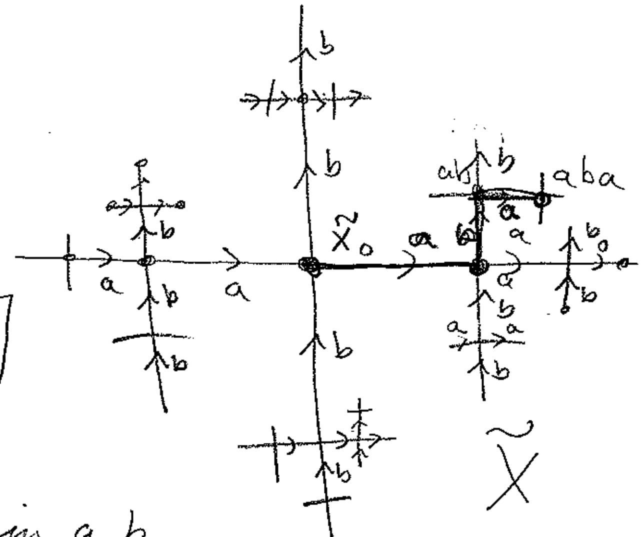
[But first, another example...]

Ex: $X = S^1 \vee S^1$



$\xleftarrow{\;p\;}$

[p defined via the labels]

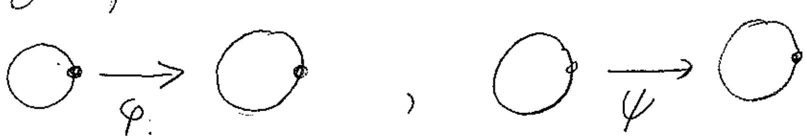


- Vertices in $\tilde{X} \longleftrightarrow$ reduced words in a, b, i.e. elts of Free Group$(a,b) = F$
- $\pi_1 \tilde{X} = 1$
- lifting bijection is a hom so $\pi_1(X, x_0) = F$.

Def: $\varphi : (Y, y_0) \to (Z, z_0)$ a map. The induced hom $\varphi_* : \pi_1(Y, y_0) \to \pi_1(Z, z_0)$ is given by $\varphi_*([f]) = [\varphi \circ f]$.

[φ_* tells us about how the topology of Y is sent to the topology of Z.]

Ex:



$\varphi(z) = 1$ $\qquad$ $\psi(z) = z^2$

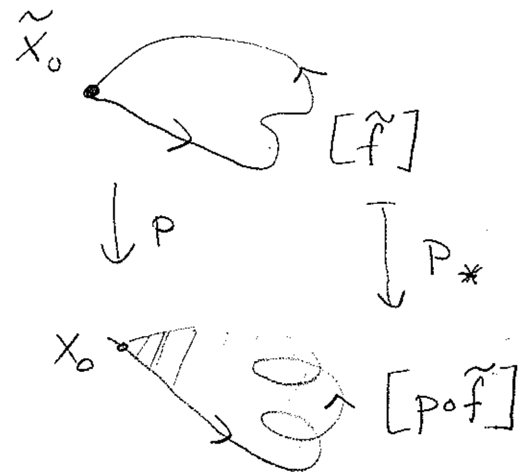
[Notice that the 2nd example is 1–1.]

[$\pi_1 S^1 = \mathbb{Z}$]

Query: $\varphi_* : n \mapsto 0$ $\qquad$ Query: $\psi_* : n \mapsto 2n$

Prop: Let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering map. Then $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is injective.

Pf: Suppose $[\tilde f]$ is in the kernel. Then $p \circ \tilde f \simeq_p \text{const}_{x_0}$. By last time, $\tilde f \simeq_p \text{const}_{\tilde x_0}$, i.e. $[\tilde f] = 1$. ▨



__Note:__ $p_*(\pi_1(\tilde X, \tilde x_0))$ = loops at x_0 which lift to loops at $\tilde x_0$.

Thm: $(\tilde X, \tilde x_0) \xrightarrow{p} (X, x_0)$ a cover w/ $\tilde X$ and X path connected. Then

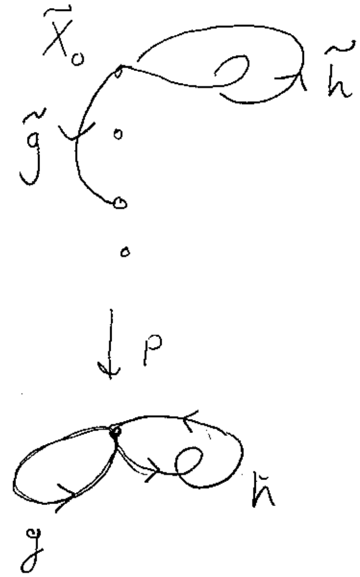
$$\pi_1(X, x_0) \Big/ p_*(\pi_1(\tilde X, \tilde x_0)) \xrightarrow[\text{ccr }\phi]{\text{lifting}} p^{-1}(x_0) \text{ is a bijection}$$

More after next theorem.

Actually didn't get to it.

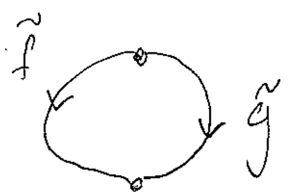
__Ex:__ $p: S^1 \to S^1$, $z \mapsto z^n$. $p_*(\pi_1 S^1) = n\mathbb{Z}$ which has index n in $\mathbb{Z}$

$p^{-1}(1) = n$ points.

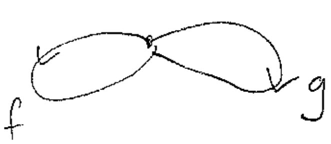


Pf: Let $H = p_*(\pi_1(\tilde X, \tilde x_0))$. As $\phi(hg) = \phi(g)$ for $h \in H$, ϕ gives a well def map on right cosets. [As argued last time,] $\tilde X$ path conn $\Rightarrow$ ϕ is onto.

Finally, suppose $\phi(f) = \phi(g)$ then fg^{-1} lifts to a loop based at $\tilde x_0$



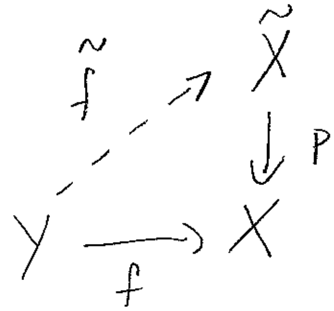
$\Rightarrow fg^{-1} \in H \Rightarrow Hf = Hg$. ▨



[__Will show:__ For reasonable spaces, given $H \leq \pi_1 X$, $\exists!$ cover w/ $p_*(\tilde X) = H$]
($\uparrow$ essentially)

Thm: $p : (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$ a cover

$f : (Y, y_0) \longrightarrow (X, x_0)$ a map



Suppose Y is path conn, locally path connected. Then f lifts to $\tilde{f} : (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$
$\iff f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$

__loc. path conn:__ $\forall y \in Y$, $\forall$ nbhds U of y, $\exists$ a path conn open set V w/ $y \in V \subseteq U$.

__Non ex:__  [Query: what point violates]

__Pf:__ $(\Rightarrow)$

$$\begin{array}{ccc} & & \pi_1(\tilde{X}_0, \tilde{x}_0) \\ & \overset{\tilde{f}_*}{\nearrow} & \downarrow p_* \\ \pi_1(Y, y_0) & \xrightarrow[f_*]{} & \pi_1(X, x_0) \end{array}$$

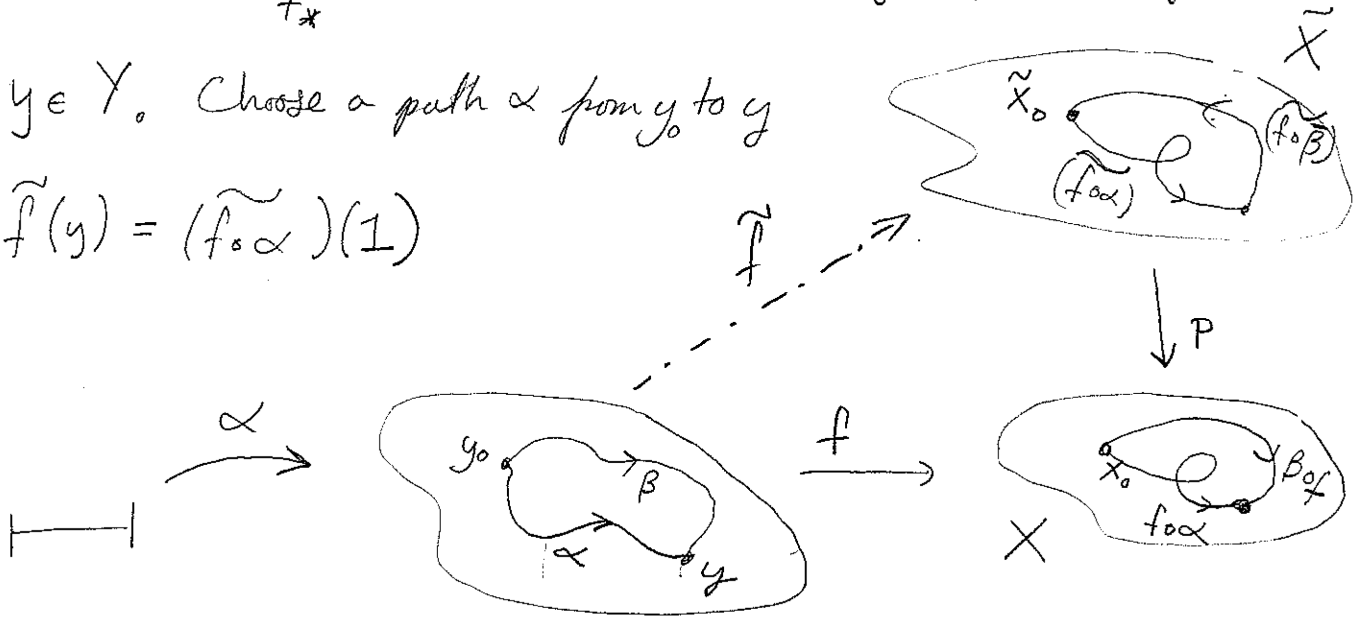
$p \circ \tilde{f} = f$, so

$p_* \circ \tilde{f}_* = f_*$

$\Rightarrow \text{image}(f_*) \subseteq \text{image}(p_*)$.

$(\Leftarrow)$ $y \in Y$. Choose a path α from y_0 to y

Set $\tilde{f}(y) = (\widetilde{f \circ \alpha})(1)$



[Idea: $\widetilde{f \circ \alpha} = \tilde{f} \circ \alpha$ assuming $\tilde{f}$ exists.]

Claim: indep of α. Suppose β is another path

Now $\alpha \cdot \overline{\beta} \in \pi_1(Y, y_0)$. By hyp, $f_*(\alpha \cdot \overline{\beta}) \subseteq \operatorname{im}(p_*) \Rightarrow$

$f \circ (\alpha \cdot \overline{\beta})$ lifts as a loop based at $\tilde{x}_0 \Rightarrow (\widetilde{f \circ \alpha})(1) = (\widetilde{f \circ \beta})(1)$.

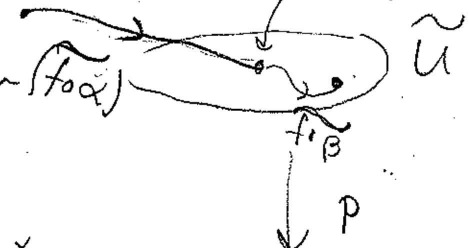
[$\tilde{f}$ is clearly a lift.]

Claim: $\tilde{f}$ is cont. Choose: U an evenly covered nbhd of $f(y)$

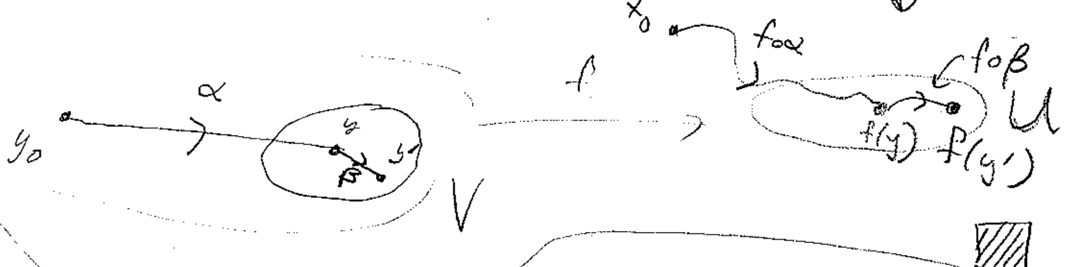
V a path conn nbhd of y s.t. $f(V) \subseteq U$.

Let $\tilde{U}$ be the slice of $p^{-1}(U)$ containing $\tilde{f}(y)$. Then

$\tilde{f}\big|_V = \left(p\big|_{\tilde{U}}\right)^{-1} \circ f\big|_V$ and hence $\tilde{f}$ is cont.



Consider discussing uniqueness of lifts if time permits.



Lecture 5

Last time:

$$\begin{array}{ccc} & & (\tilde{X}, \tilde{x}_0) \\ & \overset{\exists \tilde{f}?}{\nearrow} & \downarrow p \text{ covering map} \\ (Y, y_0) & \xrightarrow{f} & (X, x_0) \end{array}$$

If Y is path conn + locally path conn then $\tilde{f}$ exists iff $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$

Today and Friday: Classify all covering spaces of a reasonable space [In terms of subgroups of $\pi_1 X$]

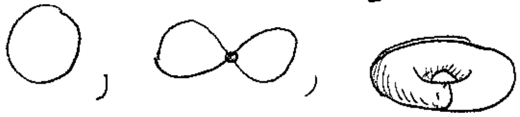
Addendum: [Lift is unique if it exists — recall that in the proof we worked via path lifting to construct $\tilde{f}$]

$$\begin{array}{ccc} & \overset{\tilde{f}_1, \tilde{f}_2}{\nearrow} & \tilde{X} \\ & & \downarrow p \text{ cover} \\ Y & \xrightarrow{f} & X \end{array}$$

Suppose Y is path conn. If $\tilde{f}_1$ and $\tilde{f}_2$ are lifts which agree at one pt, then they are equal.

Q: When does X have a simply connected cover?

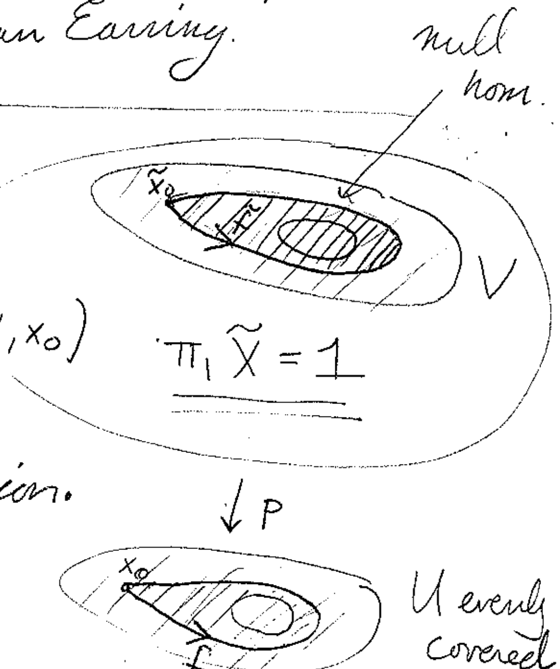
[universal cover — the biggest possible in a sense to be discussed.]

Yes:  [Q: What is the cover?]

No:  $H = \bigcup_n$ circle of diam. $1/n$ "Hawaiian Earring."

Q:  ?

(1) U evenly covered. Then $i_*(\pi_1(U, x_0)) \subseteq \pi_1(X, x_0)$ is just $\langle 1 \rangle$ where $i : U \hookrightarrow X$ is inclusion.



(2) No open nbhd of p in H has this property $\Rightarrow$ can't have a universal cover.

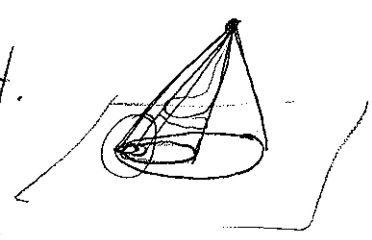
Def: X is semi-locally simply connected if every $x \in X$ has a nbhd U s.t. $\operatorname{Image}(\pi_1(U, x) \to \pi_1(X, x)) = 1$

not right? should be more like loc path conn.

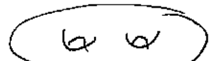
Locally simply connected means can find U w $\pi_1(U, x) = 1$

Ex:  is loc. simply connected.

Query: SLSC but not LSC? Ans: Cone on H.

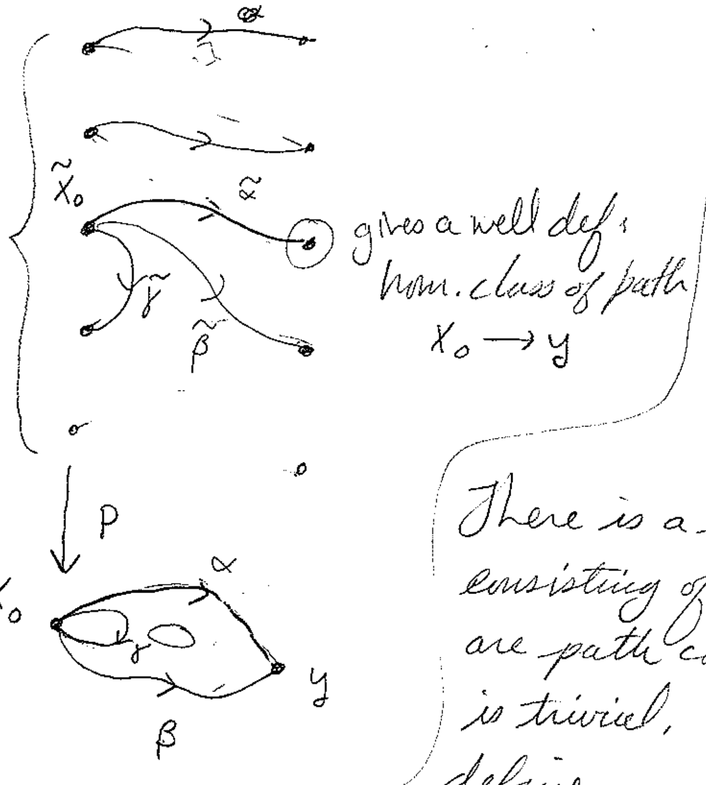


Thm: Suppose X is path conn, locally path connected and semi-loc. simp conn. Then X has a universal cover.

[Hence  has a univ. cover.]

Idea:

$p^{-1}(x_0) \cong \pi_1(X, x_0)$



Pf Sketch:

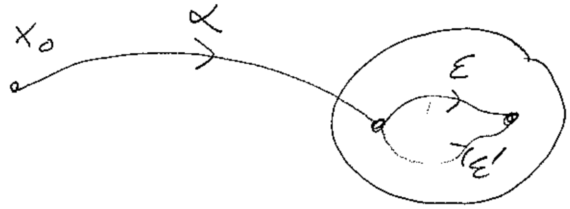
$\tilde{X} = \{ [\alpha] \mid \alpha \text{ a path starting at } x_0 \}$

$\downarrow p \qquad p([\alpha]) = \alpha(1).$

X

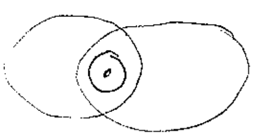
There is a basis $\mathcal{B}$ for the top of X consisting of open sets U which are path conn and $\pi_1 U \to \pi_1 X$ is trivial. For such a U define

$U_{[\alpha]} = \{ [\alpha \cdot \varepsilon] \mid \varepsilon \text{ a path contained in } U \}$



Note: $U_{[\alpha]} \xrightarrow{p} U$ is a bijection

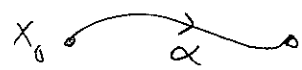
Give $\tilde{X}$ the top w/ basis $\{ U_{[\alpha]} \mid U \in \mathcal{B}, [\alpha] \in \tilde{X} \}$

[Need to check is a basis ]

For $U \in \mathcal{B}$ then $p^{-1}(U) = \coprod U_{[\alpha]}$ where $[\alpha]$ is a path from x_0 to a fixed y in U.

Hence p is cont. Moreover

$p|_{U_{[\alpha]}} \to U$ is a homeo, so with p a covering map.

Finally, $\tilde{X}$ is simply connected as 

1) $\tilde{X}$ is connected. [$\cdot [\alpha_t]$ gives a path from $[1]$ to $[\alpha]$]

2) $\pi_1 \tilde{X} = 1$ as every non-trivial loop lifts to a non-loop

$\Rightarrow P_*(\pi_1 \tilde{X}) = 1 \Rightarrow \pi_1 \tilde{X} = 1.$ ▨

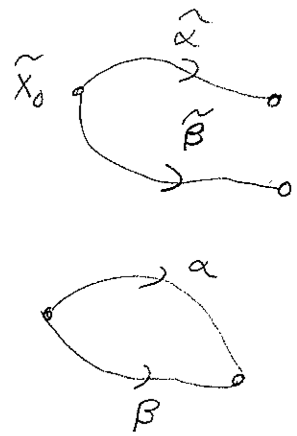
[Nice existence proof, but in practice one uses other methods.]

Thm: X as before. Given $H \leq \pi_1(X, x_0)$, $\exists$ a covering space

$$p : (\tilde{X}, \tilde{x}_0) \to (X, x_0) \quad \text{s.t.} \quad p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = H.$$

Pf: Take $\tilde{X} = \{[\alpha] \mid \alpha \text{ a path from } x_0\}/\sim$ $\quad [\alpha] \sim [\beta]$ if $[\alpha \cdot \bar{\beta}] \in H$

$\tilde{x}_0 = [\text{const}_{x_0}]$



Thm: $(\tilde{X}, \tilde{x}_0) \xrightarrow{p} (X, x_0)$ a cover with $\tilde{X}$ and X path connected. Then

$$\pi_1(X, x_0) \big/ P_*(\pi_1(\tilde{X}, \tilde{x}_0)) \xrightarrow[\text{lifting corresp.}]{\phi} p^{-1}(x_0) \text{ is a bijection.}$$

Ex: $p : S^1 \to S^1$, $z \mapsto z^n$ $\quad P_*(\pi_1 S^1) = n\mathbb{Z}$ which has index n in $\mathbb{Z}$

$p^{-1}(1) = n$ points.

Pf: [Let $H = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$. As $\phi(hg) = \phi(g)$ for $h \in H$ do have the claimed map. $\tilde{X}$ path conn $\Rightarrow \phi$ is onto.

Finally suppose $\phi(f) = \phi(g)$ then fg^{-1} lifts to a loop based at $\tilde{x}_0 \Rightarrow fg^{-1} \in H \Rightarrow Hf = Hg$.

Next time: Uniqueness of covers.

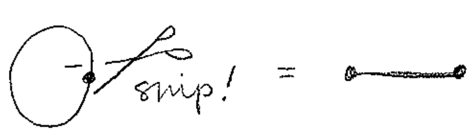
Lecture 6

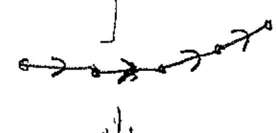
Last time: X path conn, loc path conn, semilocally simply connected. Given $H \leq \pi_1(X, x_0)$ $\exists$ a cover $p: \tilde{X} \to X$ s.t. $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = H$.

Today: Constructing covers in practice, uniqueness.

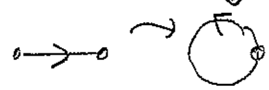
Building the universal cover: [Cut space into simply connected pieces, assemble in simply connected form.]

Ex:

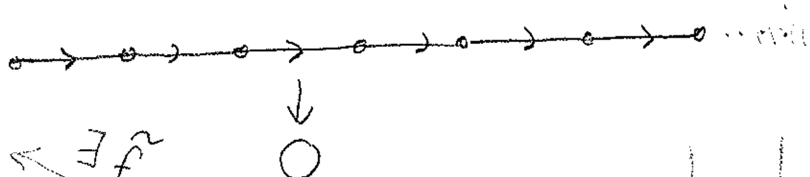




[By path lifting, for any cover the interval lifts]

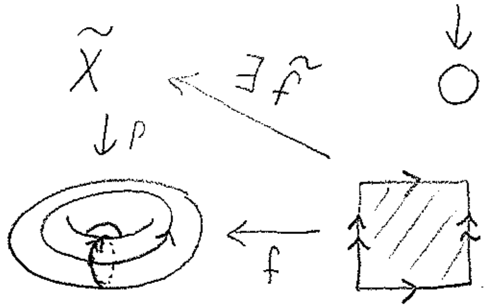


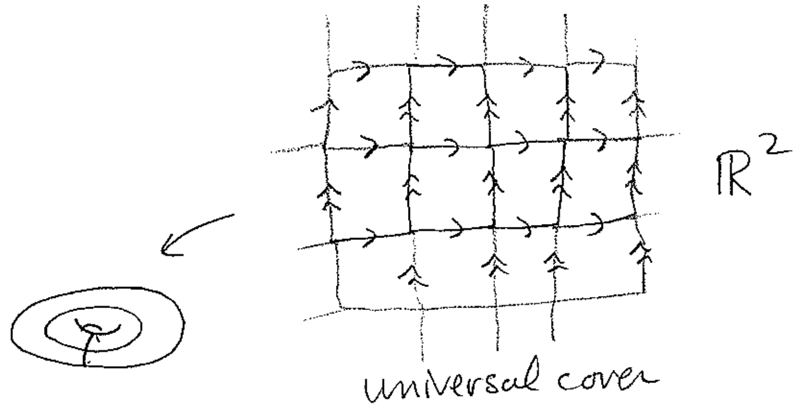
Univ cover...



[avoid unness. identifications]

Ex:



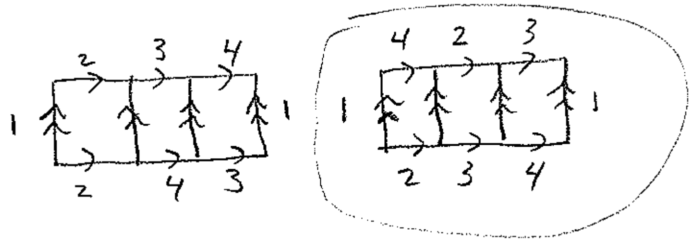


Also:

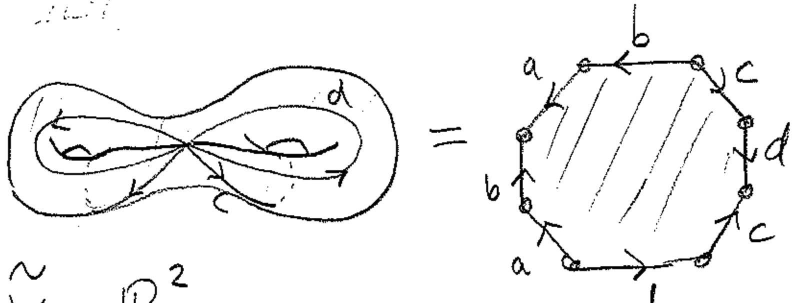
$S^1 \times S^1 \xleftarrow{p \times p} \mathbb{R} \times \mathbb{R}$

Q: [What do I need to check to make sure this is a cover?]

One these is a cover, which?



Ex:



$\tilde{X} = \mathbb{R}^2$

Handout a copy of the hyperbolic plane

[Now on to uniqueness]

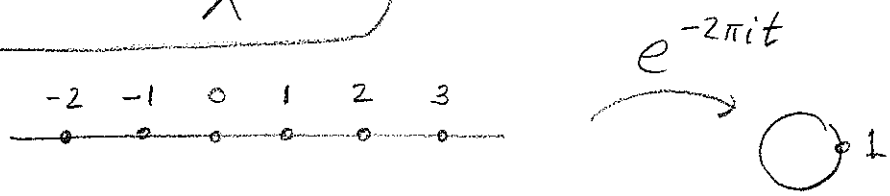
<u>Def</u>: Covers $\tilde{X}_1, \tilde{X}_2$ of X are isomorphic if $\exists$ a homeo

$$\begin{array}{ccc} \tilde{X}_1 & \xrightarrow{f} & \tilde{X}_2 \\ & {\scriptstyle p_1}\searrow \quad \swarrow{\scriptstyle p_2} & \\ & X & \end{array}$$

such that $p_2 \circ f = p_1$.

When $\tilde{X}_1 = \tilde{X}_2$ such isomorphisms are called <u>covering transformations</u>.

<u>Ex</u>:



<u>Cover trans</u>: Fix $k \in \mathbb{Z}$ $\quad f(t) = t + k$

<u>Note</u>: If f, g are two covering trans of a path conn cover $\tilde{X}$ which agree at some pt then they are equal [By uniqueness of lifts]

$$\begin{array}{ccc} \tilde{X} & \overset{f}{\underset{g}{\rightrightarrows}} & \tilde{X} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p} & \\ & X & \end{array}$$

Thus these are all of them. Note $S^1 = \mathbb{R}/\text{covering trans.} \cong \pi_1 S^1$

<u>Prop</u>: X path connected, locally path conn. Then 2 path conn covering spaces

$$\begin{array}{ccc} (\tilde{X}_1, \tilde{x}_1) & \overset{f}{\dashrightarrow} & (\tilde{X}_2, \tilde{x}_2) \\ & {\scriptstyle p_1}\searrow \quad \swarrow{\scriptstyle p_2} & \\ & (X, x_0) & \end{array}$$

are equivalent via a isom f with $f(\tilde{x}_1) = \tilde{x}_2 \iff$

$$p_{1*}(\pi_1(\tilde{X}_1, \tilde{x}_1)) = p_{2*}(\pi_1(\tilde{X}_2, \tilde{x}_2))$$

Pf: $(\Rightarrow)$ $Im(p_{1*}) = Im(p_{2*} \circ f_*) = Im(p_{2*})$.
(f_*: bijection)

$(\Leftarrow)$ $(\tilde{X}_1, \tilde{x}_1) \underset{\tilde{p}_2}{\overset{\tilde{p}_1}{\rightleftarrows}} (\tilde{X}_2, \tilde{x}_2)$, $p_1: (\tilde{X}_1, \tilde{x}_1) \to (X, x_0)$, $p_2: (\tilde{X}_2, \tilde{x}_2) \to (X, x_0)$

$\tilde{p}_2 \circ \tilde{p}_1$ a cover. trans of $\tilde{X}_1$

[Query] $\tilde{p}_2 \circ \tilde{p}_1(\tilde{x}_1) = \tilde{x}_1$

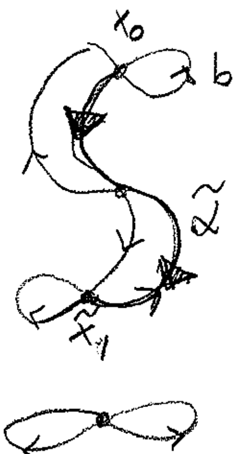
$\Rightarrow \tilde{p}_2 \circ \tilde{p}_1 = Id_{\tilde{X}_1}$

Simy $\tilde{p}_1 \circ \tilde{p}_2 = id_{\tilde{X}_2}$. So $\tilde{p}_1$ and $\tilde{p}_2$ are inverse isomorphisms.

Thm: X path conn, loc path conn, semi-loc-simp con

$$\left\{\begin{array}{c}\text{Isom classes of}\\ \text{path conn covers}\\ p: (\tilde{X}, \tilde{x}_0) \to (X, x_0)\end{array}\right\} \xleftrightarrow{\text{bijection}} \text{Subgroups of } \pi_1(X, x_0).$$

Note: The LHS is classes of covers w/ choice of $\tilde{x}_0$ in $p^{-1}(x_0)$



$p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \ni b$ but not a

$p_*(\pi_1(\tilde{X}, \tilde{x}_1)) \ni a$ but not b

this does contain $ab^{-1}aba^{-1}$

and these groups are conj

$$p_*(\tilde{\alpha})\, p_*(\pi_1(\tilde{X}, \tilde{x}_0))\, p_*(\tilde{\alpha})^{-1} = p_*(\pi_1(\tilde{X}, \tilde{x}_1))$$

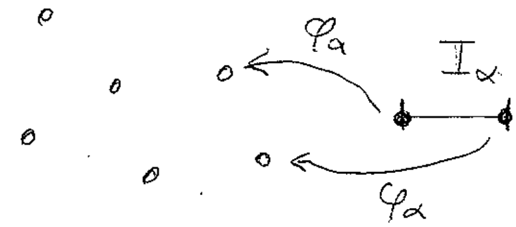
$$\left\{\begin{array}{c}\text{isom classes}\\ \text{of covers}\\ p: \tilde{X} \to X\end{array}\right\} \xleftrightarrow{\text{bijection}} \left\{\begin{array}{c}\text{conj classes of}\\ \text{subgps}\end{array}\right\}$$

Lecture 7

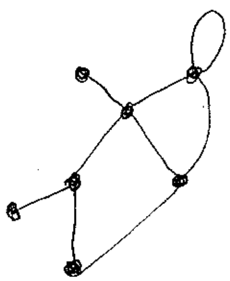
Last time: $\left\{\begin{matrix}\text{path con} \\ \text{covers} \\ \text{w/ basept}\end{matrix}\right\} \xleftrightarrow{\text{bijection}} \left\{\begin{matrix}\text{subgroups} \\ \text{of } \pi_1(X, x_0)\end{matrix}\right\}$

Today: Graphs and CW complexes, $\pi_1(\text{Graph}) = \text{free gp}$ (Sect 1.A)
: homotopy equivalences.

Graph: vertices: X^0 a discrete set of points



edges: $\coprod_\alpha I_\alpha$, $\varphi_\alpha : \partial I_\alpha \longrightarrow X^0$

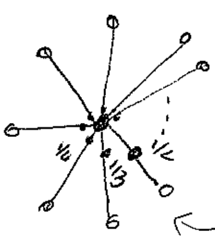


$$X = X^0 \amalg \left(\coprod_\alpha I_\alpha\right) \Big/ x \sim \varphi_\alpha(x) \text{ for } x \in \partial I_\alpha$$

with the quotient topology [i.e. set is open in X iff its preimage is open in $X^0 \amalg (\coprod_\alpha I_\alpha)$]

draw first.

[Note: vertices can have ∞ valence]



[Query:] Does this sequence converge?

A: No – [so this isn't a metric topology.]

← unit intervals [Not an issue when the valence is finite]

draw large

[Generalization:] CW Complex [Gen simplicial / Δ complexes.]

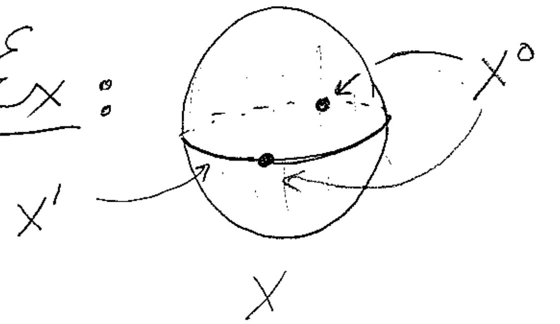
X^n – built from cells of dim $\leq n$, $D_k = \{x \in \mathbb{R}^k \mid \|x\| \leq 1\}$

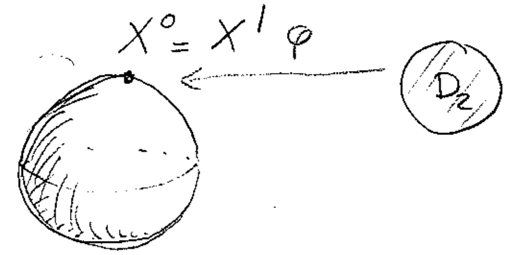
$$X^{n+1} = X^n \coprod_\alpha D_\alpha^{n+1} \Big/ x \sim \varphi_\alpha(x) \text{ for } x \in \partial D_\alpha^{n+1}$$

$\varphi_\alpha : D_\alpha^{n+1} \longrightarrow X$ cont [attaching maps.]

[X^0 pts, X^1 a graph...]

Ex:

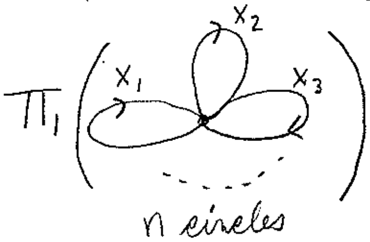




Note: can have cells of all dimensions. [See ch 0 for more.]

[Now: π_1(Graph)] —— // —— // ——

Thm: Suppose X is a graph. Then $\pi_1(X)$ is a free group.

Ex: $\pi_1\Big($  $\Big)$ = Free group on n-gens.

[e.g. words in $x_1^{\pm 1}, \dots, x_n^{\pm 1}$ with only the ob. cancel.]

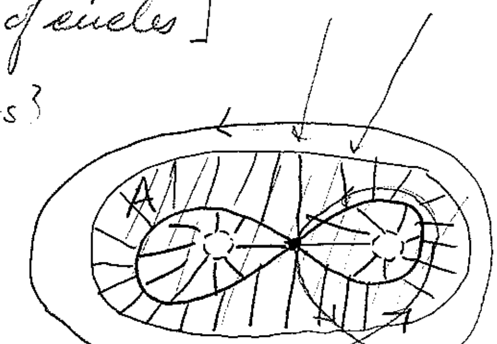
Cor: Every subgroup H of a free group F is also free.

Pf: Let X be a conn graph with $\pi_1(X) = F$. [As X is loc. path conn and locally contractible] By last week, $\exists$ a cover $p: \tilde{X} \to X$ corresponding to H. Now $\tilde{X}$ is also a graph, so $H \cong \pi_1 \tilde{X}$ is free ▨

↑ [think of edges as paths and lift]

[Idea behind thm, reduce to case of a wedge of circles]

What does the fund. gp see? $X = \mathbb{R}^2 \setminus \{\text{two pts}\}$
$A = S^1 \vee S^1$



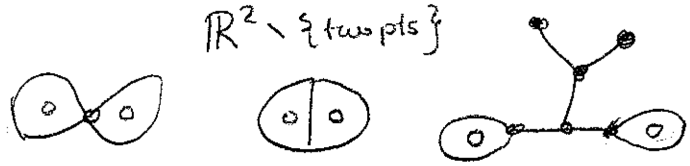
Def: A deformation retraction from X to A is a homotopy $f_t : X \times I \to X$ s.t. $f_0 = id_X$, $f_1(X) = A$ and $f_t|_A = id_A$ for all t.

Prop: If $x \in A$, then $\pi_1(A, x) \xrightarrow{i_*} \pi_1(X, x)$ is an isomorphism.

Pf idea: i_* is onto as can collaspe any loop onto A
is 1-1 for similar reason.

[only if asked $\pi_1(A,x) \underset{i_*}{\overset{f_{1*}}{\rightleftarrows}} \pi_1(X,x)$ blah blah...]

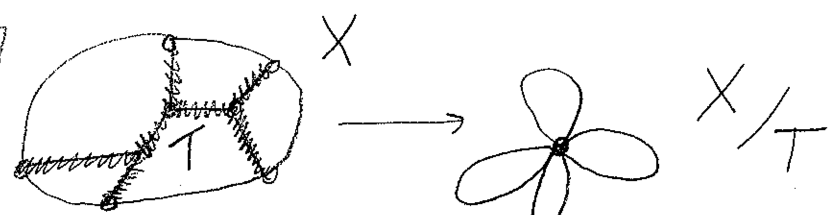
Thus all all o the these have $\pi_1 = F_2$:



Def: Spaces X and Y are <u>homotopy equivalent</u> if $\exists\, X \underset{g}{\overset{f}{\rightleftarrows}} Y$ such that $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$.

If f is a hom equiv, then $f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, f(x_0))$ is an isomophism. (Say why if we ignore basepts, see prop 1.19 for details.) For more on hom equiv see Ch 0.

[Returning to graphs]



<u>Tree</u>: A conn. graph w/ no cycles.

[T def retracts to a pt]

<u>Lemma</u>: T a tree. Then T is <u>contractible</u>, i.e. homotopy equiv to a pt.

Pf: For a finite tree use induction. In gen, use a straight line homotopy.

<u>Fact</u>: X a top space, T a clsd subset of X which is contractible. If T is "reasonable" then $X \to X/T$ is a homotopy equiv.

Thm: X a conn. graph. Then $\pi_1 X$ is free.

(↓ be fluent with this)

Pf: By Zorn's Lemma, X contains a maximal tree T which must contain every vertex. Then $X/T = \bigvee_{e_\alpha} S^1$

where e_α is an edge of X not in T. So $\pi_1 X = \pi_1 X/T$

$=$ free gp. ■

[End with a discussion of changing perspective in math.]

Lecture 8

Last time: $\pi_1(\text{Graph}) =$ free group ← [was needed for the proof of]

Today: $A \subseteq X$ "reasonable", A contractible. Then $X \longrightarrow X/A$ is a homotopy equiv.

"reasonable" = homotopy extension prop.

Def: A space A is contractible if it is homotopy equivalent to a pt. [hard to see w/ pres. tech.]

Equiv, $id_A \simeq$ (Const map $A \mapsto a_0$ [a pt in A)

Ex: $\mathbb{R}^n$, a tree  *Non Ex*:  or any $S^n \subseteq \mathbb{R}^{n+1}$ even though $\pi_1 = 1$.

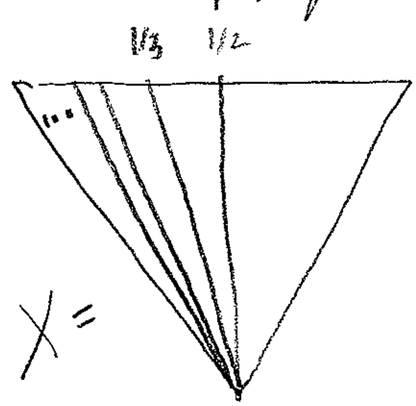
$S^\infty = \bigcup_{n=1}^{\infty} S^n$ [HW!]

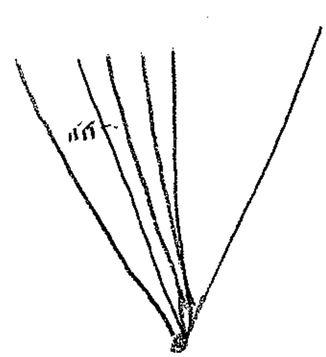
(↓ just so quotient is Hausdorff)

Suppose A is a contractible closed subset of X. When is $X \to X/A$ a homotopy equiv?

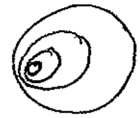
Not always:

Draw large!

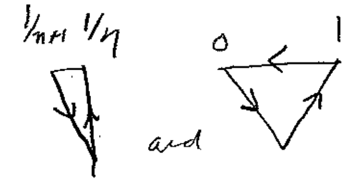
$X =$ 

$A =$ 

[Query:] What is X/A? [It's a space we've seen before.]

A: Hawaiian earring  So $X/A \not\simeq X$ as

$\pi_1 X/A$ uncountable, but $\pi_1 X$ is countable, gen by

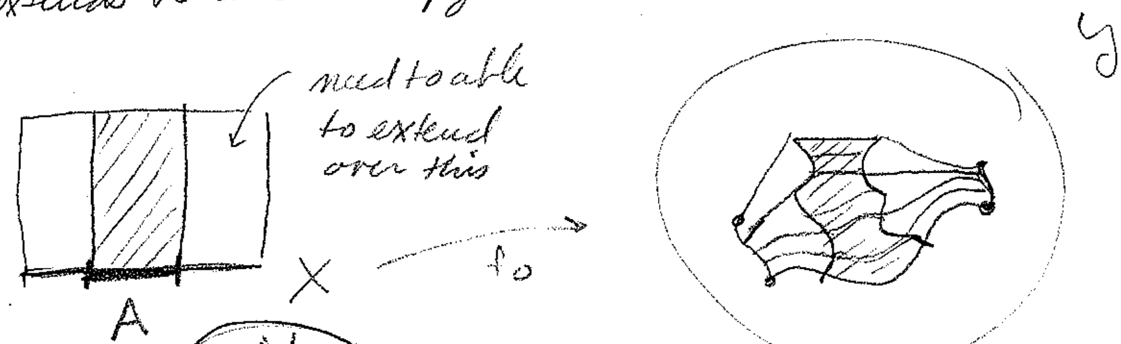


[even its abelianization is!]

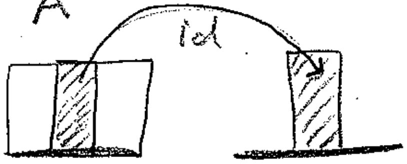
Thm: Suppose X is a CW complex, A a subcomplex, [A union of cells, itself a CW complex]
If A is contractible then $X \to X/A$ is a homotopy equiv.

[This applies to the case of a tree in a graph that we need from last time.]

Def: (X,A) have the <u>homotopy</u> <u>extension</u> <u>property</u> if given $f_0 : X \to Y$ and a homotopy $f_t : A \times I \to Y$ of $f_0|_A$. Then f_t extends to a homotopy over all of X of f_0.



Equiv cond:



$Y = A \times I \cup \{X \times 0\}$

extends to a map $r : X \times I$ to T w/ $r|_T = id_T$. [Can use r to do any ext as well!]

Say $X \times I$ <u>retracts</u> to T. [Q:] How does this diff from a def retract. [A] can delete top but not add it.

Ex: $A \subseteq X$ a subcomplex of a CW complex. [will show in a minute]

Non Ex: original example, or $X =$ 

A must be closed in X is Hausdorff, as $T = (\text{graph of } r) \cap \text{diagonal}$

only if asked →

<u>Thm</u>: If (X,A) has the homotopy extension prop then the quot map $X \xrightarrow[q]{} X/A$ is a homotopy equivalent.

<u>Pf</u>: $\exists$ a homotopy $f_t : X \times I \to X$ w/ $f_0 = id_X$, $f_t(A) \subseteq A$ and $f_1(A) = pt$.

Have

$$\begin{array}{ccc} X & \xrightarrow{f_t} & X \\ q \downarrow & & \downarrow q \\ X/A & \xrightarrow{\bar{f}_t} & X/A \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{f_1} & X \\ q \downarrow & \overset{g}{\nearrow} & \downarrow q \\ X/A & \xrightarrow[\bar{f}_1]{} & X/A \end{array}$$

g map induced by f_1

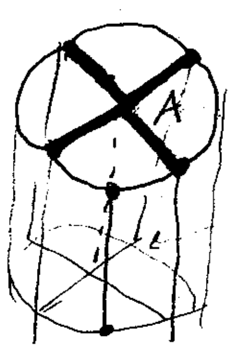
<u>Note</u>: q and g are inverse homotopy equiv: $g \circ q = f_1 \simeq id_X$

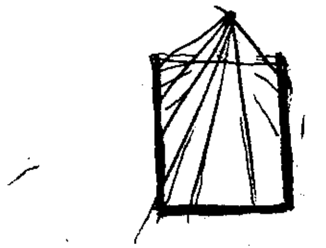
$q \circ g = \bar{f}_1 \simeq id_{X/A}$ ▨

<u>Ex</u>: X a graph, A a subgraph

Ex: X a graph A a subgraph. Does (X,A) have hom. extension?

Yes:

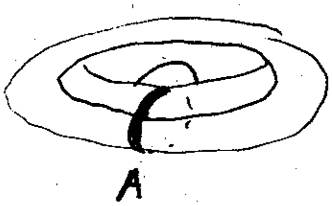


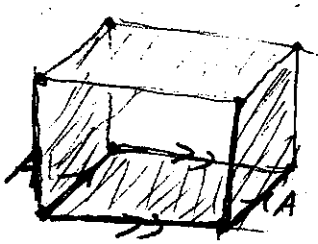


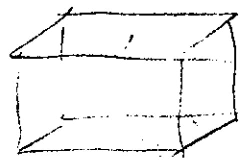
need to retract onto marked sides

annothe case

This generalizes to show a CW pair (X,A), where A is a subcomplex, has the homotopy extension prop.



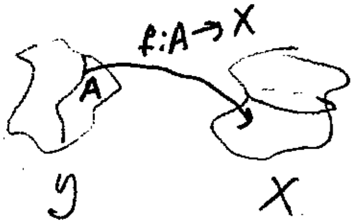




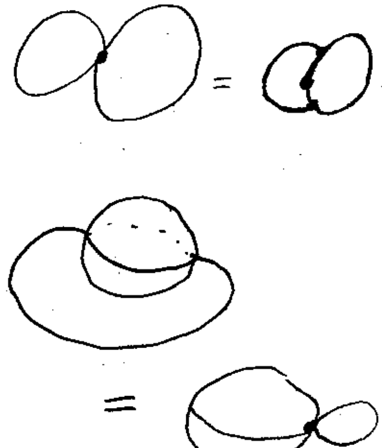
See pgs 14 – 17 for more apps, including

Thm: (X, A) has the hom. ext. prop. If $A \overset{i}{\hookrightarrow} X$ is a homotopy equivalence, then X deformation retracts to A.

Annother way to get a homotopy equivalence is to homotope an attaching map



$= Y \cup_f X = Y \sqcup X \Big/ \substack{f(a) = a \\ a \in A.}$

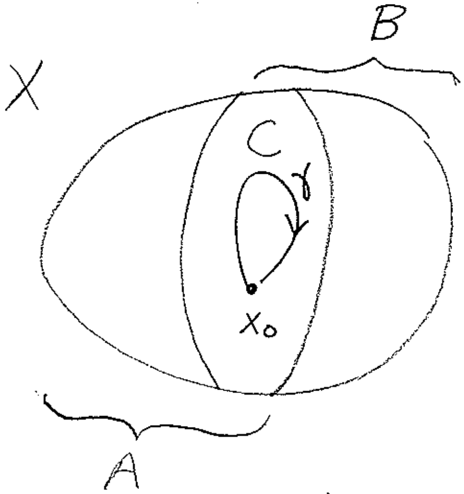


See Chap 0 pg 12 - 13 for more.

Lecture 9[redacted]

Last time: Not so relevant.

Today: Computing π_1 from pieces $\Rightarrow$ computing for CW complexes.

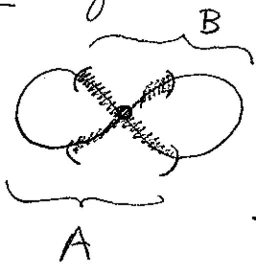


Van Kampen Thm: Suppose $X = A \cup B$ where $A, B, C = A \cap B$ are path conn open sets. For $c_0 \in C$ we have

$$\pi_1 X = \pi_1(A, c_0) * \pi_1(B, c_0) \Big/ \left\langle \{ i_{A*}(\gamma) \cdot i_{B*}(\gamma)^{-1} \mid \gamma \in \pi_1(C, c_0) \} \right\rangle$$

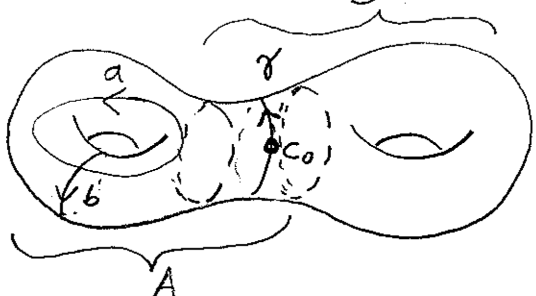
where $i_A : C \to A$ and $i_B : C \to B$ are the two inclusions.

Ex: If $\pi_1 C = 1$, then $\pi_1 X$ is just the free product.



$$\pi_1 X = \pi_1 S^1 * \pi_1 S^1 = F_2$$

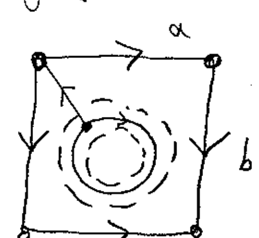
Ex:



[Query:] $\pi_1 C = \mathbb{Z}$ w/ gen γ

[Query:] $\pi_1 A = F_2$

A =

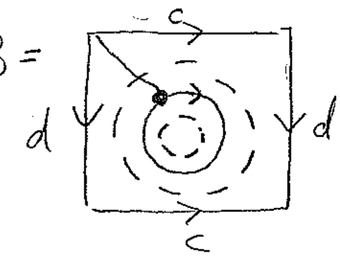


$i_A(\gamma) = aba^{-1}b^{-1}$

$i_B(\gamma) = cdc^{-1}d^{-1}$

A def retracts to

B =

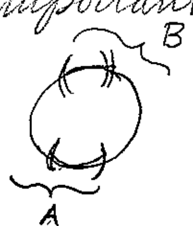


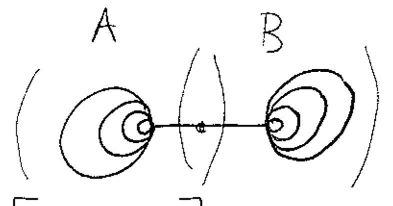
$$\pi_1(X) = \pi_1 A * \pi_1 B \Big/ \langle aba^{-1}b^{-1}(cdc^{-1}d^{-1})^{-1} \rangle$$

$$= \langle a, b, c, d \mid aba^{-1}b^{-1}dcd^{-1}c^{-1} \rangle$$

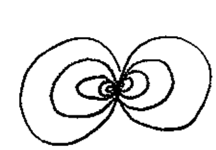
Note: Hyp are important.

[connectedness]





vs.

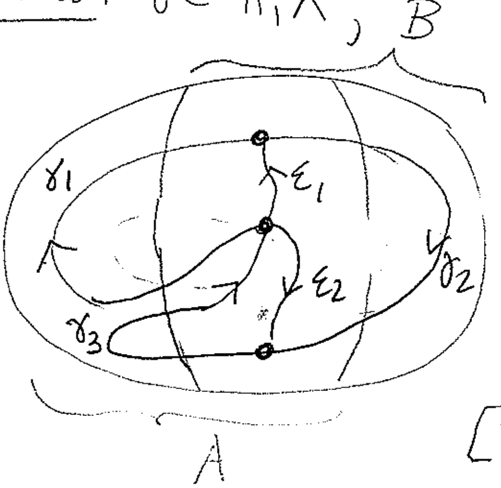


[openness]

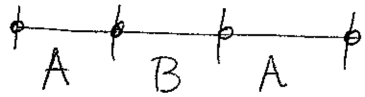
Pf: Have a hom $\pi_1 A *_{\pi_1 C} \pi_1 B \longrightarrow \pi_1 X$

↑ [notation for group in statement.]

<u>Onto</u>: $\gamma \in \pi_1 X$,



$\gamma : I \to X$. Can subdivide I into finitely many subints so each goes into A or B



$\gamma = \gamma_1 \cdot \gamma_2 \cdot \gamma_3$

$[\gamma] \simeq [\gamma_1 \cdot \bar{\varepsilon}_1] \cdot [\varepsilon_1 \cdot \gamma_2 \cdot \bar{\varepsilon}_2] \cdot [\varepsilon_2 \cdot \gamma_3]$

$\in \pi_1 A \qquad \in \pi_1 B \qquad \pi_1 A$

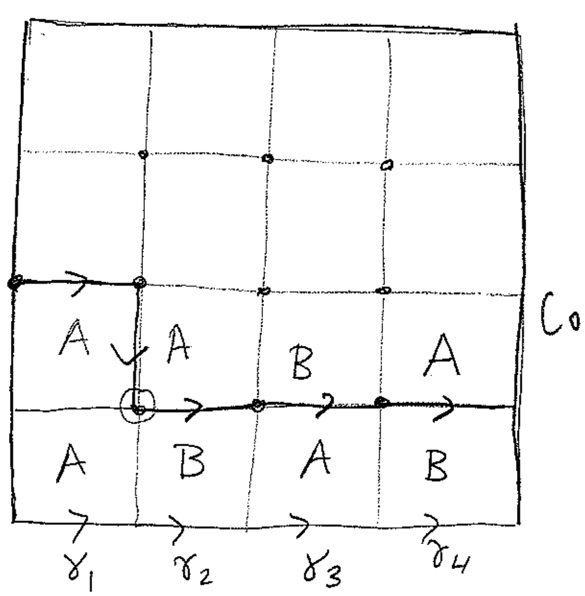
<u>1-1</u>: Suppose $W = \gamma_1 * \gamma_2 * \cdots * \gamma_n$ is in the kernel.

$\gamma_i \in \pi_1 A$ or $\pi_1 B$. ↑ [free product, not concat]

Opps not changing W in $\pi_1 A *_{\pi_1 C} \pi_1 B$: 1) If $\gamma_i, \gamma_{i+1} \in \pi_1 A$ then rep $\gamma_i * \gamma_{i+1}$ w/ $(\gamma_i \cdot \gamma_{i+1})$

2) If $\gamma_i \in i_{A*}(\gamma)$ replace with $i_{B*}(\gamma)$ $\gamma \in \pi_1 C$.

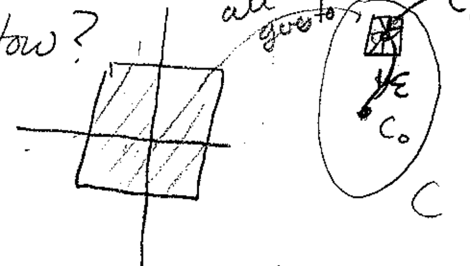
constant path at c_0

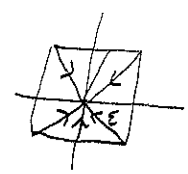


Choose small boxes so that each closed box maps into either A or B.

Modify homotopy so that all verts go to c_0 [without changing]

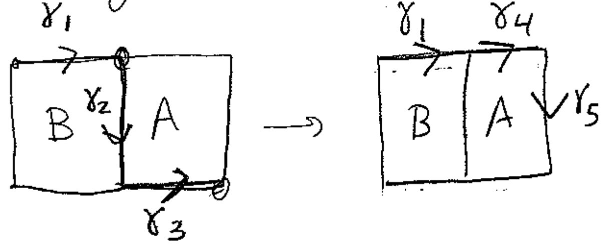
[Query!] How?





<u>Point</u>: Now any path P in grid has a corresponding word $W' \in \pi_1 A * \pi_1 B$ which maps to the geometric path P under $\pi_1 A *_{\pi_1 C} \pi_1 B$.

Now check inductively that all these paths give the same word in $\pi_1 A *_{\pi_1 C} \pi_1 B$.



$$\gamma_1 * \gamma_2 * \gamma_3 = \gamma_1 * \gamma_1' \cdot \gamma_3'$$

($\gamma_2 \in \pi_1 B$, $\gamma_3 \in \pi_1 A$, $\gamma_1', \gamma_3' \in \pi_1 A$)

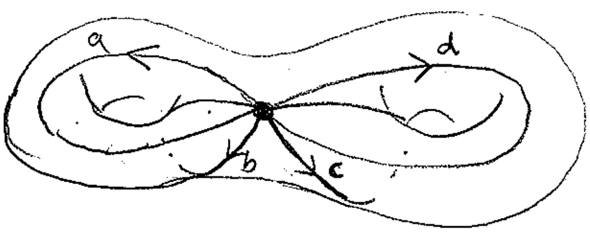
$$= \gamma_1 * \gamma_4 * \gamma_5$$

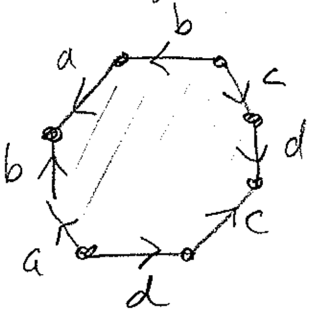
Computing π_1 of a [connected] CW complex.

Key: 1) $\pi_1(X^1) \to \pi_1(X)$ is onto.

2) $\pi_1(X^2) \to \pi_1(X)$ is an isomorphism.

Van Kampen $\Longrightarrow$ $\pi_1 X = \langle \pi_1(X^1) = \text{free} \mid \text{relations coming from the 2-cells} \rangle$





$\pi_1 = \langle a, b, c, d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1} \rangle$

Lecture 10

Last time: Van Kampen: $X = A \cup B$ with $A, B, C = A \cap B$ path conn open sets. Then

$$\pi_1 X = \pi_1 A *_{\pi_1 C} \pi_1 B = \pi_1 A * \pi_1 B \Big/ \langle i_{A*}(\gamma) = \gamma_{B*}(\gamma) \text{ s.t. } \gamma \in \pi_1 C \rangle$$

Today: π_1(CW complex). More on covering transform.

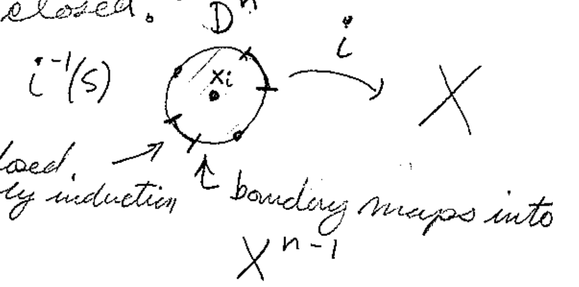
Thm: X a CW complex. Then $\pi_1(X^2) \to \pi_1(X)$ is an isom.

[Therefore, can use Van Kampen to compute π_1 of CW complex]

Pf: 1) $\pi_1 X^1 \to \pi X$ is onto. $\gamma: I \to X$ a loop based at a vertex $x_0 \in X$.

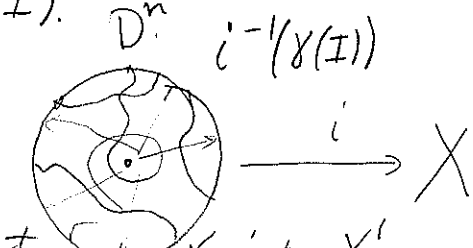
[Need that γ can be hom into X^1]. *Claim:* By compactness $\gamma(I)$ intersects the interiors of only finitely many cells of X.

Let $S = \{x_i\}$ be a set of points where the interior of of any cell in X contains at most one x_i. Then S is closed.



Any subset of S is also closed, so the top on S is discrete. As a discrete compact set is finite this proves the claim.

Consider a max dim'l cell intersecting $\gamma(I)$.



Perturb γ inside this cell so that $\gamma(I)$ misses the center. Now push out of $i(\text{int}(D^n) \setminus \{0\})$. Inductively, can homotope γ into X^1.

2) $\pi_1 X^2 \to \pi_1 X$ is injective: if γ is hom to const in X, can push hom into X^2 in the same way. ▨

Cor: $\pi_1 S^n = 1$ for $n > 1$. [Has a CW decomp w/ one 0-cell and one n-cell.]

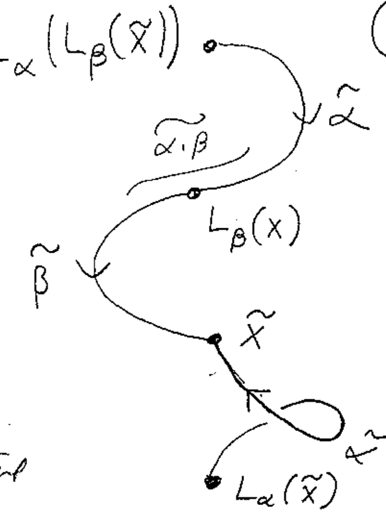
These ideas + HEP prove CW Approx Thm: $f: X \to Y$ a map of CW complexes. Then f is homotopic to g s.t. $g(X^n) \subseteq Y^n$.

[Will use a lot next quarter.]

Return to covering spaces:

$p: \tilde{X} \to X$ a cover, $\alpha \in \pi_1(X, x_0)$. Define $L_\alpha \in \text{Sym}(p^{-1}(x_0))$ by
$$L_\alpha(\tilde{x}) = \tilde{\alpha}(0) \text{ where } \tilde{\alpha} \text{ is the lift of } \alpha \text{ ending at } \tilde{x}.$$

Key: $L_{\alpha\cdot\beta} = L_\alpha \circ L_\beta$ $\quad L_{\alpha\cdot\beta}(\tilde{x}) = L_\alpha(L_\beta(\tilde{x}))$

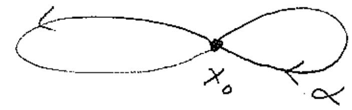


and so L_α really is a bijection and

$L : \pi_1(X, x_0) \to \mathrm{Sym}(p^{-1}(x_0))$ is a homomorphism

[Query:] $\mathrm{Stab}(\tilde{x}) = p_*(\pi_1(\tilde{X}, \tilde{x}))$

[Query:] $\tilde{X}$ is connected $\iff$ the action is transitive



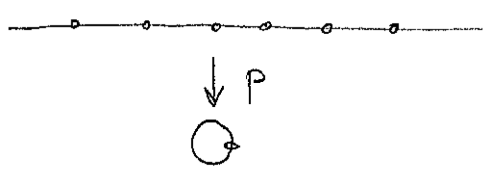
Thm: X path conn, loc path conn, semi loc. simp conn.

$$\left\{\begin{matrix}\text{connected covers of} \\ X \text{ w/ } n\text{-sheets} \\ \text{w/o base pt}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix} L : \pi_1 X \to S_n \\ \text{w/ trans. image} \\ \text{mod conj in } S_n\end{matrix}\right\}$$

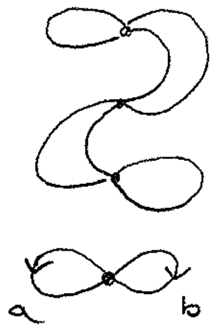
Recall: Covering transformation $\tilde{X} \xrightarrow{f} \tilde{X}$ a homeo w/ $p \circ f = p$

$$\begin{matrix} \tilde{X} & \xrightarrow{f} & \tilde{X} \\ & p \searrow & \swarrow p \\ & X & \end{matrix}$$

$G(\tilde{X})$ = group of such $\left[\begin{matrix}\text{op. is} \\ \text{comp}\end{matrix}\right]$

Ex: 

$G(\tilde{X})$ = trans w/ integer shifts

Ex: 

[Query] $G(\tilde{X}) = \{id\}$

Def: A connected cover is normal, or regular, if $\forall \tilde{x}_0, \tilde{x}_1 \in \tilde{X}$ with $p(\tilde{x}_0) = p(\tilde{x}_1)$, $\exists f \in G(\tilde{X})$ with $f(\tilde{x}_0) = \tilde{x}_1$. [Ex: $\mathbb{R} \to S^1$]

Note: For reasonable X, such f exists $\iff p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = p_*(\pi_1(\tilde{X}, \tilde{x}_1))$

$$= \gamma \cdot p_*(\tilde{X}, \tilde{x}_0)\gamma^{-1}$$

Thm: X path conn, loc path conn, SLSC. Then a connected cover $\tilde{X} \xrightarrow{p} X$ is regular $\iff p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ is a normal subgroup of $\pi_1(X, x_0)$

[What is the quotient? $G(\tilde{X})$.]

[(I ran out of material in this lecture, add more in for next time. Could have mentioned any group is π_1(CW complex) for instance.]

Lecture 11

Important note: No class friday, see email for details.

Last time: Covering transformations $\tilde{X} \xrightarrow{f} \tilde{X}$, f homeo, $p = p \circ f$

$G(\tilde{X})$ = group of such

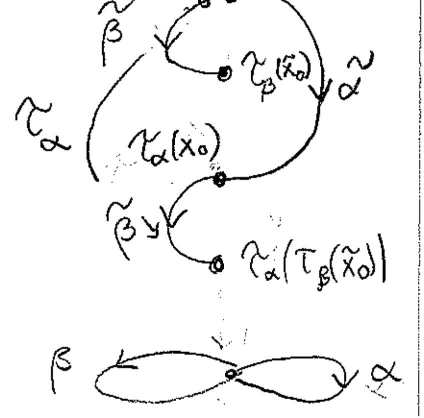
Def: A path conn cover $\tilde{X}$ is *regular* if $\forall \tilde{x}_0, \tilde{x}_1 \in \tilde{X}$ w/ $p(\tilde{x}_0) = p(\tilde{x}_1)$, $\exists f \in G(\tilde{X})$ w/ $f(\tilde{x}_0) = \tilde{x}_1$.

Thm: X reasonable. Then a conn. cover $\tilde{X} \xrightarrow{p} X$ is regular $\iff$ $p_*(\pi_1(\tilde{X}))$ is a normal subgp of $\pi_1(X)$. // *Today:* Covering spaces, the thrilling conclusion.

[What is the quotient group $\pi_1(X) / p_*(\pi_1(\tilde{X}))$?]

$(\tilde{X}, \tilde{x}_0) \xrightarrow{p} (X, x_0)$ a regular cover. Given $\alpha \in \pi_1(X, x_0)$, let $\tau_\alpha \in G(\tilde{X})$ be the unique element s.t. $\tau_\alpha(\tilde{x}_0) = \tilde{\alpha}(1)$ where $\tilde{\alpha}(0) = \tilde{x}_0$

Note: $\tau_{\alpha \cdot \beta} = \tau_\alpha \circ \tau_\beta$



and

$\tau_\alpha = id \iff \alpha \in p_*(\pi_1(\tilde{X}, \tilde{x}_0))$

[Query:]

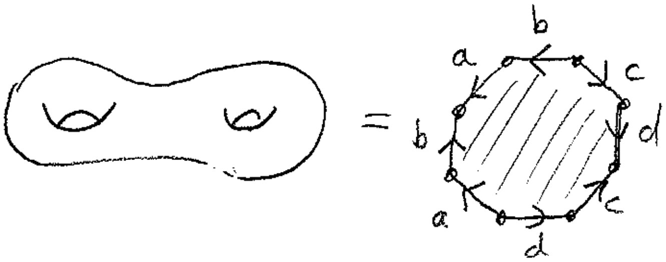
<u>Thm</u>: $\tilde{X}$ a path conn <u>regular</u> cover of X.

$$\tau: \pi_1(X,x_0) / p_*(\pi_1(\tilde{X},\tilde{x}_0)) \longrightarrow G(\tilde{X}) \text{ is an isomorphism.}$$

<u>Pf</u>: <u>Onto</u>: $\tilde{X}$ path conn. 1-1: observation above. $\blacksquare$

<u>Ex</u>: $\tilde{X}$ the universal cover of X, get $\pi_1(X,x_0) = G(\tilde{X})$ as $\langle 1 \rangle$ is normal.

and $X = \tilde{X} / G(\tilde{X}) = \pi_1(X,x_0)$. Eg: $S^1 = \mathbb{R}/\mathbb{Z}$,  $= \mathbb{R}^2/\mathbb{Z}^2$



has universal cover shown in handout. [Discuss]

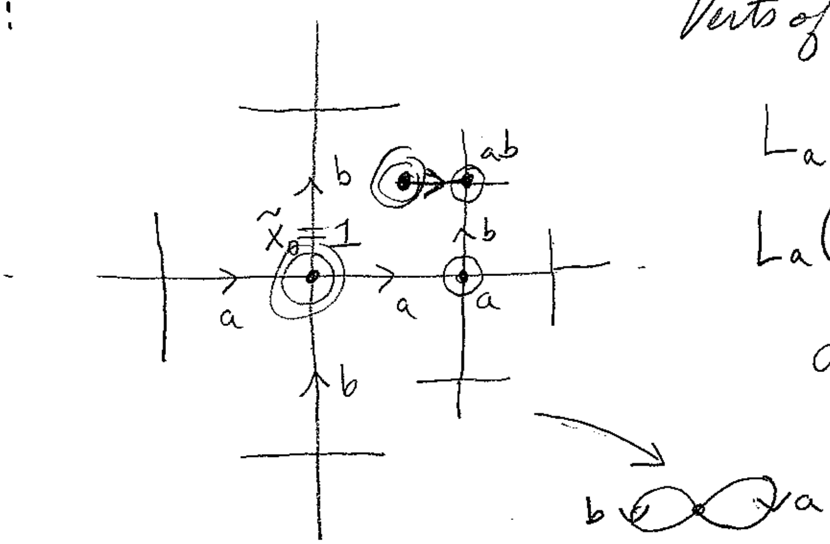
Also $H \le \pi_1(X,x_0)$, then the cover corresponding to H is $\tilde{X}/\tau(H)$

<u>Note</u>: $\tau: \pi(X,x_0) \longrightarrow G(\tilde{X})$ gives an action of π_1 on $p^{-1}(x_0)$.

This is <u>not</u> the lifting action from last time. [where "not" = "has nothing to do with".]

$L_\alpha(\tilde{x}) = \tilde{\alpha}(0)$ where $\tilde{\alpha}(1) = \tilde{x}$ 

<u>Ex</u>:



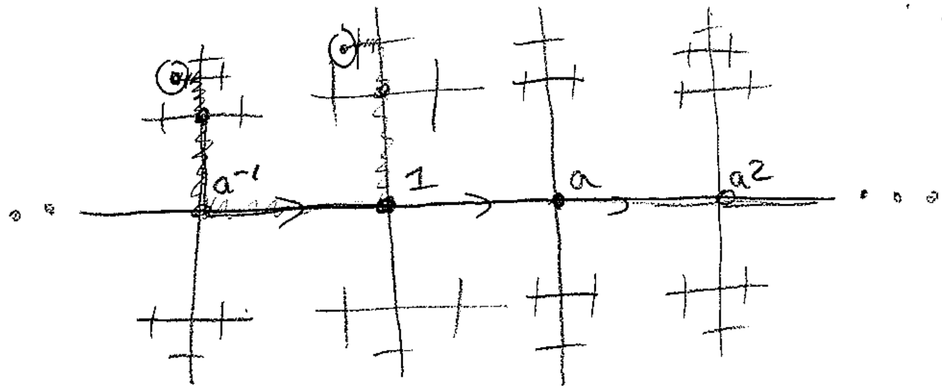
Verts of $\tilde{X} \longleftrightarrow w \in F_2$

$L_a(ab) = aba^{-1}$ $\quad L_a(a) = 1$

$L_a(w) = wa^{-1}$

$d(L_a(w), w) = 1 \quad \forall w.$

Does not extend to a homeo.



$\cdots\ \tau_a(1) = a$

simply translates along — axis

$\tau_a(a^{-1}b) = b$

$\tau_a(w) = aw$.

$d(\tau_a(w), w)$ can be very large

<u>Cayley Graph</u>: G group. $X_G = \begin{cases} \text{vertices} = \text{elts of } G \\ \text{edge between } g_1 \text{ and } g_2 \iff \\ g_1 = g_2 s^{\pm 1} \quad s \in G \end{cases}$

S a gen set

Have action of G on the left pres. this action. [Discourse on word hyperbolic groups]

<u>Constructing covers by quotients</u>: $\tilde{X}$ simply connected, G a gp of homeos of $\tilde{X}$. <u>Q</u>: When is $\tilde{X} \longrightarrow \tilde{X}/G$ a cover

[quotient top.]

<u>Need</u>: 1) G acts freely, i.e. $g(\tilde{x}) = \tilde{x} \Rightarrow g = id$. [prop of cov. trans]
<u>at Least</u>

2) Orbits don't accumulate: $G = \mathbb{Q}$ acting on $\tilde{X} = \mathbb{R}$ by translations

X, $\mathbb{R}/\mathbb{Q}$ has the top where only open sets are ϕ, X so $\tilde{X} \to X$ is not a cover.

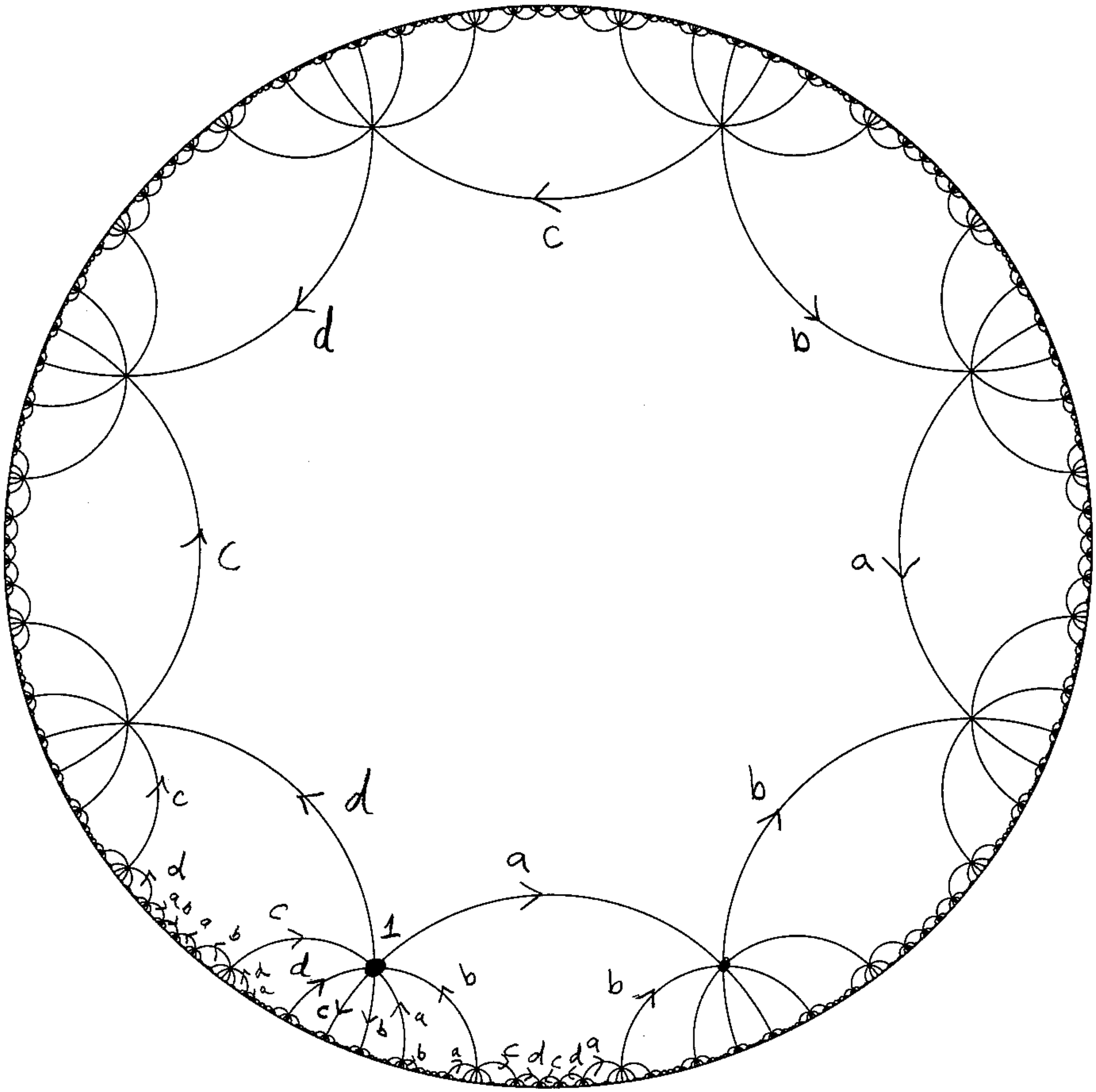
[Right condition: $\forall\, \tilde{x} \in \tilde{X}$ has a nbhd U s.t. all images $g(U)$ for varying G are disjoint.]

<u>Ex</u>: G finite [See HW.]

In practice, often construct covers this way.

$SL_2\mathbb{Z} \leq SL_2\mathbb{R}$ $\qquad$ $SL_2\mathbb{R} \to SL_2\mathbb{R}/SL_2\mathbb{Z}$ $\quad$ Modular forms, etc.

Could also discuss level 2 cong cover.



c
d
b
c
a
d
b
a
1